

Chapter IV. I must say, however, that no book is ever without some problems and this book seems far better than average. A graduate student in topology would gain a lot from reading this book and wouldn't suffer too much. He would probably need to consult some other sources, which wouldn't be hazardous to his education.

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ROBERT E. STONG

BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 1, Number 4, July 1979
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 0002-9904/79/0000-0310/\$02.50

Modern methods in partial differential equations, an introduction, by Martin Schechter, McGraw-Hill, New York, 1977, xv + 245 pp.

In the theory of linear partial differential equations, one is given an equation of the form

$$Pu = \sum_{|\alpha| < m} p_{(\alpha)}(x) D^{\alpha} u = f, \quad x \in \Omega, \quad (1)$$

generally supplemented by boundary conditions or one or more hyper-surfaces in Ω , and one asks questions about the solutions of (1), typically in one of the following three categories:

- (2) Existence.
- (3) Uniqueness.
- (4) Qualitative behavior.

The last category is quite broad; one is asking what the solutions look like. One wants to know “everything” about them, ideally; such properties as regularity, propagation of singularities, and estimates in various norms are special cases, but of course endlessly more questions arise, such as behavior of nodal sets, decay of solutions, location of maxima, limiting behavior under (possibly quite singular) perturbations of the equation or the boundary, spectral behavior of P , and many more.

The best hold on solutions to (1), giving one the best chance to say everything about the solutions, is obtained in situations where one can write out the solution explicitly in terms of special functions of hypergeometric type. Attempts to do this make up a large fraction of the story of partial differential equations up to the early part of this century. What is involved is a separation of variables, usually based on the presence of symmetries in the problem, and this method works only on a very restricted class of operators (generally, constant coefficient operators) and for boundaries of very special and simple types. This subject, together with the Hamilton-Jacobi approach to single first order equations, and the Cauchy-Kowalevski theorem, makes up most of what one considers the “classical” methods of partial differential equations. Even though research in PDE had taken a more abstract turn by the early 1900s, with the method of integral equations and functional analysis playing an important role, textbooks on PDE seemed locked into the old methods until comparatively modern times.

The early 1960s saw a number of texts on modern PDE coming out, and among them was the beautiful book by Bers, John and Schechter [3]. One part dealt with the interior regularity of solutions to elliptic PDE's, using L^2 and Sobolev space estimates and Gårding's inequality. This material also forms the core of the book under review, but before we discuss the contents of the present book, it is a good idea to consider what the modern methods in linear PDE are, and we offer the following list of nine methods, which is not exhaustive.

I. Functional analysis. The use of various Hilbert, Banach, Fréchet, and LF spaces, and the use of operator theory, particularly on Hilbert spaces, pervades most of modern linear PDE, to such an extent that these days the use is as often implicit as explicit. Sobolev spaces and spaces of distributions form the setting for most of the analysis, though other spaces, particularly Besov spaces and various Lipschitz classes, make an occasional appearance. Within the past 10 years or so, a movement has been afoot to replace distributions by hyperfunctions and to replace functional analysis by “algebraic analysis,” when working with operators with analytic coefficients. This approach is discussed in [25].

II. Fourier analysis. The use of Fourier series and/or the Fourier transform in constant coefficient PDE is intimately connected with separation of variables, and Fourier series was used by Daniel Bernoulli in the very beginning of the subject. In the modern approach, Fourier analysis, via the Plancherel theorem particularly, is often used to get *estimates* on solutions to constant coefficient equations, obtained from variable coefficient equations by “freezing” the coefficients, and, if a boundary is present, flattening out the boundary. One then patches these estimates together to get estimates in the variable coefficient situation. Also, Fourier analysis in the complex domain has made possible much detailed information on the behavior of general classes of constant coefficient PDE's; see Ehrenpreis [7].

III. Energy estimates. Here are included L^2 estimates for solutions and their derivatives, hence Sobolev space estimates. One of the basic estimates of this

sort is Gårding's inequality,

$$\operatorname{Re}(Pu, u) \geq c_1 \|u\|_{H^m}^2 - c_2 \|u\|_{L^2}^2, \quad u \in C_0^\infty(\Omega) \quad (5)$$

where $P(x, D)$ is an operator of order $2m$ which is strongly elliptic, i.e.,

$$\operatorname{Re} P_{2m}(x, \xi) \geq c|\xi|^{2m}.$$

When P is second order, (5) is proved simply by integration by parts. For higher order operators, one freezes coefficients, obtains such an estimate as (5) for constant coefficient operators by Fourier analysis, and glues these estimates together. Energy estimates are also used to prove existence, uniqueness, and finite propagation speed for hyperbolic equations, and it is in the study of second order hyperbolic equations that the term "energy" has a most direct physical interpretation. Weighted L^2 estimates, known as Carleman estimates, have also played an important role.

IV. Fundamental solutions and parametrices. A distribution $E(x, y)$ such that

$$P(x, D)E = \delta_y \quad (6)$$

is called a fundamental solution. If the two sides of (6) differ by a smooth function, E is called a parametrix. One can get a lot of information about solutions to (1) if one of these is known. For variable coefficient equations it is usually impossible to construct a fundamental solution; one is generally happy to be able to construct a parametrix. Often for elliptic operators a first order approximation to a parametrix is obtained by freezing coefficients and dropping lower order terms and straightening out the boundary, getting fundamental solutions for such simple equations, via Fourier analysis, and gluing these together. A true parametrix is then obtained by an iterative procedure. This technique is often called the Levi parametrix method. For regular elliptic boundary value problems, this method is useful for obtaining L^p and Hölder estimates on solutions, which are not available from the energy estimate methods. For nonelliptic operators, much more subtle methods are required to construct parametrices, some of which involve tools V and VI.

When a fundamental solution of a particularly simple equation is available, it may yield a great deal of information about perturbed equations. For example, a careful analysis of the fundamental solution to the free space Schrödinger equation $\partial u / \partial t = i\Delta u$, which is simply $(4\pi it)^{-n/2} e^{-i|x-y|^2/4t}$, is one of the most powerful tools in the study of the Schrödinger equation with a nontrivial potential, $\partial u / \partial t = i\Delta u + Vu$.

V. Singular integrals, pseudo differential operators, and Fourier integral operators. Operators of these sorts are very convenient because they allow one to localize a problem in momentum space and in position space, i.e., to "microlocalize." Energy estimates can consequently be microlocalized, and regularity theorem strengthened to propagation of singularities results. Also, energy estimates can be sharpened somewhat, with such delicate tools as the "sharp Gårding inequality" and the Calderón-Vaillancourt theorem. Fourier integral operators are used to construct parametrices for hyperbolic equa-

tions, and for more general equations with simple characteristics, and this extends the classical method of geometrical optics. Fourier integral operators also play the key role in the next tool.

VI. Canonical transformations. If P is a pseudo-differential operator and J is an elliptic Fourier integral operator, JPJ^{-1} is a pseudo-differential operator whose principal symbol is obtained from that of P via a canonical transformation. By this method, one can often reduce an equation to a standard form which is easier to treat.

VII. Maximum principle and Harnack's inequality. This method is useful for second order elliptic and parabolic equations, and is particularly useful for such equations in divergence form with merely bounded measurable coefficients, a class of equations one is forced to consider in the study of quasi-linear second order elliptic equations.

VIII. Potential theory and Brownian motion. These methods grew out of an intensive study of the Laplace equation and the heat equation, and the relation between the heat equation and diffusion processes. The solution to such an equation, with Dirichlet boundary conditions on an arbitrarily nasty boundary, can be written down using the Feynman-Kac formula, which involves an integral over path space with respect to Wiener measure. The Strong Markov property allows one to localize problems, and this together with the countable additivity of Wiener measure allows for very subtle results. For Neumann boundary conditions, one analyzes reflecting Brownian motion.

IX. Symmetry properties. The use of symmetry continues to play a role in modern PDE. Most notable is the work of Folland and Stein [10], constructing fundamental solutions of "sub-Laplacians" on Heisenberg groups, and then using essentially the Levi parametrix method to construct parametrices for the Kohn Laplacian \square_b on pseudo-convex domains.

The textbook by Schechter treats existence and regularity of solutions to regular elliptic boundary value problems, with smooth coefficients and smooth boundaries, and existence and uniqueness of solutions to hyperbolic initial value problems and certain other evolution equations, essentially using methods I–III. The author gives a leisurely treatment of these topics, devoting separate chapters to interior estimates for solutions to elliptic equations (and more generally hypoelliptic equations with constant strength) with constant coefficients, and with variable coefficients, to boundary estimates in the frozen coefficient, half-space case, and to the general variable coefficient case for curved boundaries, first for the Dirichlet problem and then, in the last chapter, for general regular elliptic boundary conditions. The author gives some results on the spectrum of a strongly elliptic operator at the end of the ninth chapter of this ten chapter book, and promises the reader more information in a following chapter, but this material seems to have been omitted from the book, so we refer the reader to Agmon [1] or Schechter [26] for further information on the spectrum.

The author avows the aim to keep the analysis fairly elementary, so he uses

only spaces $L^2(\Omega)$ and the Sobolev spaces $H^m(\Omega)$ where m is a positive integer or (when Ω is a boundary) a half integer, and avoids discussing distributions. Since generalized functions more singular than elements of L^2 are not considered, the concept of a fundamental solution is not available, and it does not arise in this book. A weak solution to a PDE $Pu = f$ is defined to be an element $u \in L^2(\Omega)$ such that

$$\int u P'v = \int f v$$

for all $v \in C_0^\infty(\Omega)$, where P' is the formal transpose of P . By this definition, the equation

$$x \frac{\partial}{\partial x} u = 1 \quad (7)$$

has a weak solution, while

$$x^2 \frac{\partial}{\partial x} u = 1 \quad (8)$$

does not, though certainly (8) has a distribution solution, namely $-P.V. 1/x$.

In the first chapter, the question of local solvability of (1) is discussed. Local solvability for constant coefficient PDE's is proved. Also, the H. Lewy example of an equation with no solutions is discussed in the very beginning of the book. This discussion precedes the introduction of the concept of a weak solution, so the theorem given there merely states that, for some $f \in C_0^\infty$, the equation

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u + 2(ix - y) \frac{\partial}{\partial t} u = f \quad (9)$$

has no C^1 solution. Of course, (7) furnishes a trivial example of an equation with no C^1 solution, so many readers may wonder what all the fuss is about. However, the proof Schechter gives shows that (9) has no weak solution.

In addition to elliptic and hypoelliptic equations, whose analysis forms the central theme of the book, the author discusses hyperbolic equations, in Chapters 4 and 5, these chapters separating the interior regularity results from the study of elliptic boundary value problems. He restricts attention to the constant coefficient case, and derives global estimates. There is no discussion of finite propagation speed, nor of Huyghens' principle, nor of propagation of singularities.

Given these restrictions on the scope of the book, it seems to be an appropriate text for a year's course in PDE, at the senior undergraduate or beginning graduate level, perhaps in concert with another book dealing with a complementary set of material, such as one of the broad, elementary texts by Garabedian [11], or Courant and Hilbert [4], or maybe a book that goes more deeply into hyperbolic theory, e.g., Lax and Phillips [18]. The topics in Schechter's book are contained in Hörmander [13], where they are given a somewhat brisker treatment. Also, elliptic boundary value problems are treated in Lions and Magenes [19] in a fashion similar to that presented here.

There is not a single comprehensive treatment of all the methods of linear PDE mentioned in this review, so it would be good to point out some sources

where the student could learn about these topics. For Method I, see Yosida [29], Reed and Simon [24]. For Method II, Stein and Weiss [28]. Method III, Hörmander [13] Lax [17], Agmon [1]. Method IV, Hörmander [13] Chapter 10, Agmon, Douglis, and Nirenberg [2], Garabedian [11], Chapter 5. Method V, Stein [27], Hörmander [14], [15], Duistermaat and Hörmander [5]. Method VI, Duistermaat and Hörmander [5], Egorov [6], Sato et al. [25]. Method VII, Protter and Weinberger [23], Gilbarg and Trudinger [12]. Method VIII, Ito and McKean [16], McKean [20], Friedman [9]. Method IX, Folland and Stein [10], Miller [21].

In keeping with the limitations of the book under review, we have made no mention of techniques in nonlinear PDE. We would like to end this review with a plea for a book, by one or more experts, giving a broad and advanced treatment of this vast and woefully underdeveloped subject.

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MICHAEL E. TAYLOR

BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 1, Number 4, July 1979
 © 1979 American Mathematical Society
 0002-9904/79/0000-0311/\$01.50

Information and exponential families in statistical theory, by O. Barndorff-Nielsen, Wiley, New York, 1978, ix + 238 pp., \$33.00.

If we take n samples of a random variable which is normally distributed with unknown mean μ and known variance σ^2 , then the sample mean \bar{X} is normally distributed with mean μ and variance σ^2/n . Thus the probability that $\sqrt{n}(\bar{X} - \mu)/\sigma$ lies between -1.96 and 1.96 is, from the standard normal table, equal to $.95$ and hence it seems that the probability that μ lies between $\bar{X} - 1.96\sigma/\sqrt{n}$ and $\bar{X} + 1.96\sigma/\sqrt{n}$ is also $.95$. The latter statement, which seems to the naive reader to be equivalent to its immediate predecessor, makes no sense since the fixed number μ either lies in the given interval (a 95% confidence interval) or it doesn't and most elementary books on statistics take some pains to explain this fact. What the statement really means is that if we use this method over and over again to infer that μ is in the given interval we will be right about 95% of the time. Unfortunately this limit of frequency concept coincides with most people's intuitive idea of probability and very likely was used that way earlier in the same book. The same situation obtains in the more realistic situation of an unknown σ except that the t -distribution is used instead of the normal and many other statistical problems lead to this sort of impasse.

The apparent duality here; each choice of μ determining a distribution of \bar{X} and each sample mean \bar{X} seeming to determine a distribution of μ ; has led to a great deal of heated dispute among statisticians, who tend to be rather feisty anyhow. The first great storm center seems to have been the fertile brain of R. A. Fisher. Fisher's idea of fiducial inference is not much mentioned anymore but the debate goes on as the book under review illustrates.

The book is divided into three parts the first of which is concerned, among other things, with the above mentioned duality. In fact, x serves as the probability variable while ω is reserved for the parameter. This emphasizes the duality perhaps at the expense of occasionally confusing the over-conditioned probabilist. Barnard's notion of l.o.d.s. functions is introduced but not extensively developed. The likelihood function and Barndorff-Nielsen's somewhat controversial plausibility function seem to be the only well-known examples. Much of this part is devoted to the, more or less, dual notions of sufficiency and ancillarity. In fact four different definitions are given of each,