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BULLETIN (New Series) OF THE
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Hilbert's third problem, by Vladimir G. Boltianskiĭ (translated by Richard A. Silverman and introduced by Albert B. J. Novikoff), Scripta Series in Math., Wiley, New York, 1978, x + 228 pp., $19.95.

1. Since the response to the title of this book is invariably "What *is* Hilbert's third problem?", let us begin by considering the problem itself. Loosely speaking, it asks whether there is any way of deriving the formula for the volume of a tetrahedron without using calculus. Clearly there is no hope of avoiding all mention of limits in most questions of volume, for it is by appealing to a limit process that the very notion of volume is extended to any figure more general than a rectangular solid having rational edges. Analogously, limits are needed to extend the concept of area beyond rectangles having rational sides. Hilbert's problem acknowledges such fundamental involvement with limit processes by taking as its point of departure the formula for the volume of an arbitrary rectangular solid. The plane analogue of his question is whether the formula for the area of a triangle can be derived without calculus when the formula for the area of a rectangle is assumed. Because a triangle can always be partitioned into three pieces which can be re-assembled to form a rectangle, the plane version of the problem is easily resolved in the affirmative. However, Hilbert observed that no known derivation of the formula for the volume of a tetrahedron was able to negotiate the path beyond the formula for a rectangular solid without appealing to a limit process, and he inquired whether this was by choice or necessity. Just as plane polygons can be "triangulated", solid polyhedra can be decomposed into tetrahedra. Therefore the volume of a polyhedron reduces to the case of the tetrahedron, and Hilbert's third problem is equivalent to the question of whether an elementary theory of the volume of polyhedra is possible.

Hilbert, however, did not pose his problem in such general terms. He was very specific. Clearly two polyhedra A and B have equal volumes if it is possible to decompose them into polyhedral pieces such that the parts of A are congruent piece-for-piece to the parts of B, that is, if they are *equidecomposable*. Similarly, the volumes of A and B are equal if there exists a set of polyhedra that can be applied to each of A and B to build up the same resulting figure, that is, if A and B are *equicomplementable*. The methods of equidecomposability and equicomplementability characterize the elementary methods of dealing with volume. Suspecting a negative answer to his question, Hilbert called specifically for

> *the exhibition of two tetrahedra having equal bases and equal altitudes that can be shown to be nonequidecomposable and nonequicomplementable.*

In 1900, the very year in which Hilbert's now famous 23 research problems were put forth, Max Dehn succeeded in confirming Hilbert's suspicion. Central to his solution was the demonstration that a regular tetrahedron and a cube of equal volume are nonequidecomposable and nonequicomplementable.

2. In Euclidean 3-space, then, two polyhedra can have the same volume without being either equidecomposable or equicomplementable. Thus the concepts of equality of volume, equidecomposability, and equicomplementability are not all equivalent. (However, in 1943, the equivalence of equidecomposability and equicomplementability was established by J.-P. Sydler.)

We would not have expected this result from an investigation of the Euclidean plane, where all three concepts are, in fact, equivalent. Here the Bolyai-Gerwien theorem asserts that polygons A and B of equal area are always equidecomposable. Possible restrictions on the motions that are needed or permitted in carrying the pieces of A to their destinations in B has become a question of major interest concerning the method of equidecomposition (and equicomplementation). This constitutes a second theme which accompanies the initial problem (concerning the equivalence of the three concepts under discussion) through changing contexts due to variations not only in dimension (from the plane to n-space, $n = 3, 4, 5, \ldots$) but in the underlying geometry (Euclidean, Lobachevskian, Riemannian, Archimedean, non-Archimedean). This second line of inquiry has led to some remarkable discoveries. For example, in the Euclidean plane a polygon A can always be decomposed into pieces none of which need to be flipped over (outside the plane) in covering an equal polygon B. Furthermore, there always exists a decomposition of A each piece of which can be put into its place in B either without any turning whatsoever or after being turned exactly halfway round in the plane. A necessary and sufficient condition has even been found for the existence of a decomposition of A for which each piece can be dropped into place in B without rotations of any kind (including flipping over). Boltianskiĭ tells this story so well that one is tempted to think that pioneer work in this field is so easy and straightforward that anyone can do it.

Hilbert knew that the Archimedean character of Euclidean geometry was an essential requirement for the equivalence of equidecomposability and equality of area in the Euclidean plane. Before posing his problem, he had succeeded, by the following elegant argument, in exhibiting two non-equidecomposable triangles of equal area in any non-Archimedean geometry. Let AB and AD be two segments on a ray R whose lengths e and f, respectively, are such that $ne \geqslant f$ does *not* hold for any positive integer n. Let AC and DQ be segments of length e which are perpendicular to R. Then triangles ABC and ABQ, having the same base and equal altitudes, have the same area.

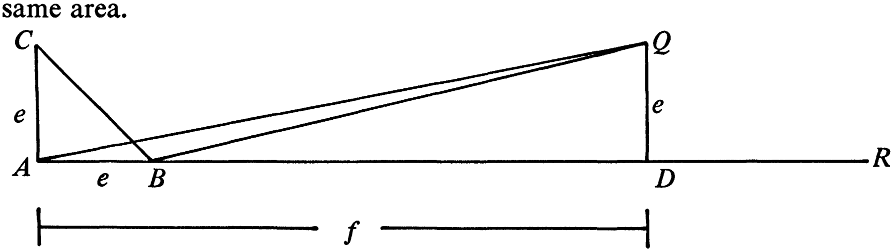

FIGURE 1

Now suppose these triangles are equidecomposable. By triangulating each polygon in the decomposition of $\triangle ABC$, and similarly triangulating the corresponding polygons in the decomposition of $\triangle ABQ$, we obtain a decomposition in which every piece is a triangle. Now, each of the k triangles in the decomposition of $\triangle ABC$ must have perimeter which does not exceed the perimeter of $\triangle ABC$ itself. By the triangle inequality, we have $BC < AC + AB = 2e$, implying the perimeter of $\triangle ABC < 4e$. Therefore the total perimeter of all the triangles in the decomposition of $\triangle ABC$ is $< 4ek$. However, for all k we have $4ek < f$, and, concerning $\triangle ABQ$, we have

$$AQ + AB = AQ + QD > AD = f,$$

giving

$$4ek < AQ + AB < \text{perimeter of } \triangle ABQ.$$

Thus, no matter how the triangles of $\triangle ABC$ are applied to $\triangle ABQ$, they do not possess enough total perimeter to cover even the boundary of $\triangle ABQ$.

3. The discoveries of the Swiss geometer Hugo Hadwiger in the 1950s brought a fresh approach and renewed interest to the elementary theory of area and volume. He and other scholars since then have succeeded in generalizing many of the results known for the plane. For example, in Euclidean 3-space, necessary and sufficient conditions have been found for two polyhedra of equal volume to be equidecomposable when there are no restrictions on the motions permitted (the Dehn-Sydler theorem, 1965), and when the motions are restricted as severely as to allow only translations (Hadwiger, 1968). If one is permitted to magnify or contract the individual pieces without changing their shapes, then any two polyhedra can be shown to be equidecomposable. In fact, if such changes in size are permitted in the plane, then, incredibly, any polygon A can be decomposed into pieces that can be made to cover any other polygon B (of whatever size and shape), where no piece is permitted to be rotated in any way whatsoever (Figure 2). Finally, let us note a remarkable inheritance of some of the polyhedra which are equidecomposable with a *cube*:

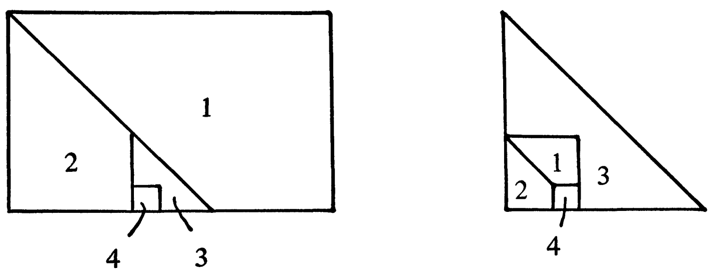

FIGURE 2

> if a polyhedron that is equidecomposable with a cube can itself be decomposed into several *congruent* polyhedra, then each of these polyhedra is also equidecomposable with a cube.

4. Such, then, are the kinds of things to be found in this book. Everything noted above, and much more, is proved in full detail with great care. The unhurried exposition is nothing if not lucid and thorough, and I would expect that a considerable number of secondary school mathematics students and teachers would find a good deal of the first half of the book (to p. 118) to be enjoyable reading at an appropriate level. The complete solution of Hilbert's third problem is contained in this part of the book and it is truly an exceptionally clever and beautiful application of elementary algebraic geometry. It is remarkable that space is so amenable to algebraic analysis.

The book begins with a 44-page treatment of the theory of area and volume from first principles. This is well done and, although mostly elementary, it goes so far as to discuss the role played by the axiom of choice and Hamel bases in establishing the independence of the axioms for area. Except for this passage, the advanced reader need only skim this first chapter. On the other hand, the beginning reader can skip this more sophisticated part without serious loss in continuity.

A second chapter of equal size is taken up with discoveries in the plane, and a 124-page final chapter carries the subject into 3-space and higher dimensions. Up to the end of the solution of Hilbert's problem, the discussion is generally easy-going and elementary. Beyond this point, the arguments soon become longer, more complicated and sophisticated (from p. 130). While the author continues to explain everything in full detail, this part of the book demands much more drive and concentration and is clearly an object for serious study. However, the motivated reader will not go unrewarded. He will discover another instance of the unity of mathematics in the way several branches of abstract mathematics converge to solve a problem of the most concrete kind. For example, the 20-page proof of the Dehn-Sydler theorem is highly algebraic and draws not only from the now standard vector methods of modern geometry (Minkowski sums) but uses techniques and results from functional equations, group and ring theory, set theory, and linear algebra. As often observed in many quarters, it is impressive what mathematics can do when it pulls itself together.

Later topics include an extension of a few of the results to spaces of higher dimension, notably 4 dimensions, and a brief discussion of connections with the modern subject known as the algebra of polyhedra. Although the volume constitutes a self-contained account of a topic which is now essentially complete, it concludes with a short list of unresolved questions.

There are a few misprints, but few mistakes that the reader will not see through in a matter of moments.

The subject is related to a surprising discovery made recently by Robert Connelly (Cornell University), who is working in this general area at the present time. In 1813, Cauchy proved that a convex polyhedron with rigid faces is itself a rigid solid, that is, even if it were hinged along every edge, its shape could not be altered without forcibly breaking the surface. Connelly produced a nonconvex rigid-faced polyhedron which, if considered to be hinged at its edges, can be moved continuously through a small range of shapes without distortion of any face. For a description of this polyhedron and instructions for constructing a model, see Robert Connelly, *A flexible sphere*, The Mathematical Intelligencer (Springer-Verlag), volume 1, number 3, 1978.

The interested reader might also be on the lookout for a forthcoming book by Irving Kaplansky on all 23 of Hilbert's Paris problems.

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Mechanizing hypothesis formation. Mathematical foundations for a general theory, by P. Hájek and T. Havránek, Universitext, Springer-Verlag, Berlin-Heidelberg-New York, 1978, xv + 396 pp., $24.00.

I know of no book on statistics that has "Hypothesis formulation" in its index; nor is it in the indexes of Ralston and Meek [17], Mathematical Society of Japan [13], Polanyi [15], Winston [20], nor Boden [1]. But some-