with this equation, which, incidentally, belongs to a type that is invariant under all coordinate transformations and hence is meaningful in principle on any differentiable manifold with boundary.

One must mention, finally, that one defect of the book which causes difficulty in the reading, is that the translation was edited very carelessly, allowing often such grammatical mistakes as inappropriate interchange between definite and indefinite articles; the term "hypersurface" is used several times in the last section in the place of "hypersphere", and the internal references are frequently inaccurate; for example, the footnote on p. 13 "See editor's note on p. 6" should apparently refer to the one actually on p. 12; on p. 102 a reference to "subsection 4", in the reviewer's opinion, apparently intends to recall material appearing in p. 73–77, which are in §5, subsection 3. Another typical, more serious inconsistency is the sentence on p. 96, "This mapping is said to be *normal.*", which would be better understood if it were worded "This mapping is called the gradient mapping." However, if we take the pragmatic view that a careful editing of the translation might have taken such a long time that the publication might have lost some of its timeliness, we may be grateful for the fact we have access in an extremely short period to a monograph which brings us essentially up to date on a beautiful subject, in which current research is active and new results are appearing very rapidly.

EUGENIO CALABI

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Accélération de la convergence en analyse numérique, by C. Brezinski, Lecture Notes in Math., vol. 584, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 295 pp., $13.70.

The late George Forsythe once described the numerical analyst as "the guy who used to be the odd man in the mathematics department and now is the odd man in the computer science department". Indeed, a person working in numerical analysis frequently is at odds with someone. Either he produces rigorous and nontrivial mathematics–in which case it often turns out that his work is of no direct use to the man in the computing center who has to put a satellite in orbit–or he creates software which really solves problems, and solves them more efficiently and more accurately than the software produced by his colleague the physicist or engineer who also dabbles in computing, and then it turns out that his work is based on plausibility considerations and unprovable assumptions, and that in its attention to irksome detail and to numerical mishaps it resembles a sophisticated piece of technological design much more than it resembles a piece of mathematics. (An example of numerical analysis of the first kind would be Varga's *Functional analysis and approximation theory in numerical analysis*; an example of the second kind, Shampine and Gordon's *Computer solution of ordinary differential equations*.)

Brezinski's numerical analysis definitely is of the mathematical kind; however, it is analysis that can be of fairly direct use also in the computation laboratory. The problem, of course, is fundamental; it might even be called *the* fundamental problem of numerical computation. Suppose we wish to compute a quantity q that is mathematically well defined, such as the value of a solution of a differential equation at a given point. Unless q is rational, which is unlikely, there is no chance that q can be computed in a finite number of rational operations, which are the only operations that are possible on a computer. The best the numerical analyst then can do is to invent an algorithm that generates a sequence $\{q_n\}$ that can be proved to *converge* to q. If he is very conscientious, he will try to estimate $|q_n - q|$ for each n. More likely than not, this effort will be regarded with ridicule by the computing community, because it will turn out that these error estimates, which must take into account the worst possible case, are too high by several orders of magnitude. The man who pays for the computing time will be more interested in the speed of convergence. To him it makes a difference whether, say, the solution of Poisson's equation for the unit cube, using 100 grid points in each direction, takes several years, as it will when Gaussian elimination is used, or a matter of seconds, which is the time required by a judicious application of the Fast Fourier Transform.

Speeding up the convergence of a slowly converging sequence thus can save dollars. But how can the speed-up be achieved? The basic principle is this: Use an approximate knowledge of the qualitative behavior of the error to approximately eliminate the error. Here is a simple example. Suppose we wish to compute $e = 2.718\ldots$ as $\lim q_n$ where

$$q_n := (1 + 1/n)^n, \qquad n = 1,2,\ldots.$$

Suppose also that the nth powers have to be computed by accumulating factors, so that the work required to compute q_n is roughly proportional to n. (There are generalizations of our simple example to the solution of initial value problems where these assumptions are appropriate, and where the analog of $e = \Sigma(n!)^{-1}$ is unavailable.) Everybody knows that the q_n converge to e extremely slowly. To compute e with an error $< 10^{-9}$, one would have to take $n > 10^9$, which even on a fast computer takes 20 minutes. But how does the error behave? It is not hard to show that the q_n admit an asymptotic expansion of the form

$$q_n \approx e + \sum_{k=1}^{\infty} c_k n^{-k} \qquad (n \to \infty). \tag{A}$$

The coefficients c_k, although mathematically well defined, are somewhat awkward to compute for a person untrained in computational analysis; as far as our problem is concerned, they even must be regarded as inaccessible, because they involve the very quantity e which we are trying to compute. But the mere fact that (A) holds with some c_k permits us to apply the Romberg algorithm, which consists in successively eliminating the c_k and which in this example yields the limit e correctly to 9 digits using merely the values q_{2^k} for $k = 1,2,3,4,5$.

Brezinski does discuss the Romberg algorithm, but the main thrust of his exposition is devoted to the Padé table which emerges as the unifying concept behind a very large number of acceleration algorithms. The transformation of Shanks, Wynn's ε-algorithm and Rutishauser's *qd* algorithm all emerge as special or limiting cases of suitable quantities of the Padé table, and so do many less well-known modifications of these algorithms. The Padé table, in turn, is closely related to (or, if viewed properly, identical with) the sequence of continued fractions corresponding to the remainders of a formal power series. There also is a "topological" form of the ε-algorithm, and there are confluent forms of it that deal with the problem of predicting $\lim f(t)$ as $t \to \infty$ through real (and not merely integer) values.

Brezinski gives a full formal treatment of all of these topics. For some of the more difficult analytical results (such as the convergence of continued fractions corresponding to moment sequences) he refers to other literature. Indeed his bibliographic documentation is very thorough. One could wish for a richer numerical documentation of the various algorithms presented. In these days of the programmable pocket calculator it is easy even for a very book-minded author to construct telling numerical examples.

The emphasis on formal relationships is likely to disappoint the pure mathematician who is always looking for depth. Maybe the time has come to state quite unabashedly the importance in applied mathematics of understanding formal relationships as well as underlying theory. The Fast Fourier Transform, whose impact on applied mathematics is already comparable to that of the differential calculus or of the method of least squares, has no "depth"; it is merely an extremely clever arrangement of certain arithmetical operations that are required in discrete Fourier analysis. It seems of some significance that, although the need for efficient discrete Fourier analysis was pressing since the advent of the electronic computer, it took 20 years to discover the " fast " Fourier transform, and that the discovery was finally made by a statistician and an electrical engineer, and not by a specialist in analytical Fourier theory.

So far, we have mainly described the theoretical numerical analyst's reaction to Brezinski's book. What about the practitioner? It lies in the nature of things that, in a way, Brezinski cannot escape from that perennial curse afflicting all of applied mathematics: The nicer the mathematical model, the farther it is usually removed from physical reality. In the daily life of computing, the situations where all hypotheses guaranteeing the performance of one of Brezinski's more elaborate algorithms can be verified will be few and far between. This should not keep the practitioner from trying acceleration algorithms also in circumstances where the hypotheses are not verifiable. In fact, as reported in G. A. Baker's *Essentials of Padé approximants* (Academic Press, 1975), Padé approximants are successfully used by physicists in scattering theory in precisely such situations. Brezinski's book will enable them to arrive at a better understanding of the analytical and formal machinery supporting these calculations.

PETER HENRICI