10. L. Kronecker, *Über die Charakteristik von Functionen-Systemen*, Monatsber, Kgl. Preuss, Akad. Wiss. 1878, 145–152.

11. J. Leray and J. Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **51** (1934), 45–78.

12. N. G. Lloyd, *On analytic differential equations*, Proc. London Math. Soc. **30** (1975), 430–444.

13. R. D. Nussbaum, *On the uniqueness of the topological degree for k-set-contractions*, Math. Z. **137** (1974), 1–6.

14. ______, *Generalizing the fixed point index*, Math. Ann. **228** (1977), 259–278.

15. P. H. Rabinowitz, *A note on topological degree for holomorphic maps*, Israel J. Math. **16** (1973), 46–52.

16. J. T. Schwartz, *Compact analytic mappings of B-spaces and a theorem of Jane Cronin*, Comm. Pure Appl. Math. **16** (1963), 253–260.

17. ______, *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.

18. J. R. L. Webb, *Remarks on k-set-contractions*, Boll. Un. Mat. Ital. **4** (1971), 614–629.

19. G. T. Whyburn, *Topological analysis*, Princeton Univ. Press, Princeton, N. J., 1958.

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The Minkowski multidimensional problem, by Aleksey Vasil'yevich Pogorelov, translated by V. Oliker, with an introductory comment by L. Nirenberg, Wiley, New York, Toronto, London, Sydney, 1978, v + 106 pp., $13.75.

The book under review is, to the reviewer's knowledge, the first exposition in English of an important topic in geometry since Busemann's text *Convex surfaces* (Interscience, 1958). It is hoped that this review, as well as Nirenberg's *Introductory commentary* which prefaces the English translation, may help popularize this beautiful subject in the English reading mathematical community.

The Minkowski problem, in its original formulation [1],[1] deals with the determination of a closed, convex hypersurface F in euclidean n-space, in terms of a given, positive valued function $f(\xi)$ ($\xi = (\xi_1, \ldots, \xi_n)$, $\sum_i \xi_i^2 = 1$) defined on the unit hypersphere S^{n-1}, where $f(\xi)$ represents the reciprocal of the Gaussian curvature of F at the point where the outward unit normal is the vector ξ. The function f (which we call the Minkowski data) must necessarily satisfy the exactness condition expressed by the vector equation

$$\int \xi f(\xi)\, d\omega(\xi) = 0, \tag{1}$$

the integration being meant over the sphere S^{n-1}.

This problem was solved originally by Minkowski only in the following, "weak" sense: given the Minkowski data satisfying (1), there exists a closed, convex hypersurface F, unique up to a translation, such that, for any given, closed region $G \subset S^{n-1}$ the integral

$$\int_G f(\xi)\, d\omega(\xi)$$

[1]References in square brackets are in terms of the bibliography at the end of Pogorelov's book.

equals the area of the region of F whose spherical image under the Gauss map (generalized, if F is not differentiable) is G. The weak solution exists in the same sense for the Minkowski problem extended by replacing the regular measure density $f(\xi)d\omega(\xi)$ on S^{n-1} by any nonnegative Borel measure whose support is not contained in any hyperplane section of S^{n-1}. On the other hand, the question naturally presents itself, how smooth is the weak solution F of the Minkowski problem in terms of any given degree of smoothness of the data $f(\xi)$. This is the *regularity problem* of the Minkowski problem, whose solution may be regarded as the principal part of Pogorelov's book, even though only eighteen pages of the book are devoted to the topic itself. It should be emphasized that the regularity problem, originally solved by Nirenberg for surfaces in $\mathbf{R}^3$ [6], was originally solved by Pogorelov himself in $\mathbf{R}^n$ for $n \geqslant 3$.

The importance of the Minkowski problem and its solution is to be felt both in differential geometry and in elliptic partial differential equations, on either count going far beyond the impact that the literal statement superficially may have. From the geometric view point it is the Rosetta Stone, from which several other related problems can be solved. Some, included in Pogorelov's book deal with the determination of closed, convex hypersurfaces by the data of other curvature functions expressed in terms of the unit normal; other generalizations deal with curvature functions expressed in terms of both the normal and the position of the point. In another direction, not mentioned by Pogorelov, the Minkowski problem can be equivalently restated in terms of affine invariants of closed, convex hypersurfaces (i.e., their determination in terms of their affine co-normal indicatrix); the solution helps in dealing with other problems in affine and centro-affine differential geometry.

From the analytical viewpoint, the Minkowski problem is interpreted as determining a convex hypersurface expressed nonparametrically as the graph of a convex function

$$x_{n+1} = z(x_1, \ldots, x_n)$$

(the shift in dimension from $\mathbf{R}^n$ to $\mathbf{R}^{n+1}$ occurs in the book as well), satisfying an equation of the type

$$\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right) = \Psi\left(\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right), \tag{2}$$

where Ψ is a given, positive valued function related to the Minkowski data $f(\xi)$. By means of a Legendre transformation (in a geometric context this amounts to describing F by means of the support function), the above equation can be transformed into the following one (using the same symbols in a different role),

$$\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right) = \Phi(x_1, \ldots, x_n), \tag{3}$$

where Φ is a given positive function, and the unknown $z(x_1, \ldots, x_n)$ is restricted by the requirement of being a convex function.

Both (2) and (3) are special cases of Monge-Ampère equations in n variables; consequently one section of the book is devoted to proving the regularity of certain boundary value problems for elliptic (i.e., convex function) solutions of (3). The concluding section of the book then returns from analysis to geometry and reproduces the author's original, ingenious proof that the only entire, strictly convex functions in $\mathbf{R}^n$ that are solutions of (2) or (3) with the right-hand term equal to a positive constant, are quadratic polynomials; the geometrical meaning of this statement is that the only (complete) improper affine (convex) hyperspheres are paraboloids.

The book furnishes an excellent illustration of the symbiotic relation between geometry and partial differential equations. A very large part of the material presented recapitulates results of Pogorelov's own original contributions, most of them less than ten years old. The reading, however, may prove difficult for the nonspecialist, because the exposition is extremely terse; the reader seeking to learn the subject might find it advantageous to begin by reading the first two chapters of Busemann's *Convex surfaces* (already quoted). For additional background material we may suggest also A. D. Alexandrov (Aleksandrov)'s book *Die innere Geometrie der konvexen Flächen* (Berlin, Akademic-Verlag, 1955), or Pogorelov's *Extrinsic geometry of convex surfaces* (AMS Translation, 1970; original Russian edition, 1969).

It is worthwhile mentioning that the term "Monge-Ampère" equation as used by the author refers to equation (3) of this review; therefore its specialization to the classical case of two variables constitutes a highly restricted case of the classical Monge-Ampère equation. A generalization of (3) that corresponds to the Monge-Ampère equation in two variables in its full generality, while retaining some of the features that make its study possible, is the following. Let $x_{n+1} = z(x_1, \ldots, x_n)$ denote the "unknown" function and denote the values of its first order partial derivatives $\partial z/\partial x_i$ by ξ_i $(1 \leq i \leq n)$; let (P_{ij}) and Φ be two functions of $2n + 1$ independent variables $x_1, \ldots, x_n; x_{n+1}; \xi_1, \ldots, \xi_n \in \mathbf{R}^{2n+1}$ with values respectively in the space of symmetric $n \times n$ matrices $(P_{ij} = P_{ji})$ and in the positive real numbers $(\Phi > 0)$ and assume, for technical reasons, that $\partial\Phi/\partial x_{n+1} \geq 0$ everywhere. Then the general Monge-Ampère equation should read

$$\det\left(\frac{\partial^2 z}{\partial x_i \partial x_j} + P_{ij}\left(x_1, \ldots, x_n; z(x); \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right)\right)$$
$$= \Phi\left(x_1, \ldots, x_n; z(x); \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right);$$

we may limit our consideration to elliptic solutions by allowing only solutions for which the symmetric matrix

$$\frac{\partial z}{\partial x_i \partial x_j} + P_{ij}(x; z; \nabla z)$$

is positive definite (for this purpose, in a connected domain, this condition needs to be verified only at one point). A problem for further research, suggested by reading Pogorelov's book would be to study the problems of existence, uniqueness and regularity for boundary value problems associated with this equation, which, incidentally, belongs to a type that is invariant under all coordinate transformations and hence is meaningful in principle on any differentiable manifold with boundary.

One must mention, finally, that one defect of the book which causes difficulty in the reading, is that the translation was edited very carelessly, allowing often such grammatical mistakes as inappropriate interchange between definite and indefinite articles; the term "hypersurface" is used several times in the last section in the place of "hypersphere", and the internal references are frequently inaccurate; for example, the footnote on p. 13 "See editor's note on p. 6" should apparently refer to the one actually on p. 12; on p. 102 a reference to "subsection 4", in the reviewer's opinion, apparently intends to recall material appearing in p. 73–77, which are in §5, subsection 3. Another typical, more serious inconsistency is the sentence on p. 96, "This mapping is said to be *normal.*", which would be better understood if it were worded "This mapping is called the gradient mapping." However, if we take the pragmatic view that a careful editing of the translation might have taken such a long time that the publication might have lost some of its timeliness, we may be grateful for the fact we have access in an extremely short period to a monograph which brings us essentially up to date on a beautiful subject, in which current research is active and new results are appearing very rapidly.

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Accélération de la convergence en analyse numérique, by C. Brezinski, Lecture Notes in Math., vol. 584, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 295 pp., $13.70.

The late George Forsythe once described the numerical analyst as "the guy who used to be the odd man in the mathematics department and now is the odd man in the computer science department". Indeed, a person working in numerical analysis frequently is at odds with someone. Either he produces rigorous and nontrivial mathematics–in which case it often turns out that his work is of no direct use to the man in the computing center who has to put a satellite in orbit–or he creates software which really solves problems, and solves them more efficiently and more accurately than the software produced by his colleague the physicist or engineer who also dabbles in computing, and then it turns out that his work is based on plausibility considerations and unprovable assumptions, and that in its attention to irksome detail and to numerical mishaps it resembles a sophisticated piece of technological design much more than it resembles a piece of mathematics. (An example of numerical analysis of the first kind would be Varga's *Functional analysis and approximation theory in numerical analysis*; an example of the second kind, Shampine and Gordon's *Computer solution of ordinary differential equations*.)

Brezinski's numerical analysis definitely is of the mathematical kind; however, it is analysis that can be of fairly direct use also in the computation