geometry) although a few results, which give sufficient conditions for M to be conformally flat, require instead that $\bar{M}$ have vanishing Bochner tensor. The well-organized proofs and calculations are cleanly presented in a straightforward easy-to-follow manner and, despite the many indices, are nearly always free of errors, even typographical. (Two exceptions: The proof of Theorem 4.1 of Chapter III–and its analogues in later chapters–does not make it clear whether the distribution L lives in $\bar{M}^m(4)$ or in the frame bundle of that manifold; in Chapter IV, Example 8.1 appears to contradict Proposition 10.2, but including the hypothesis $c \geqslant 1$ fixes it up.)

The organization of the book is straightforward and enhances its role as a reference work. Chapters I and II constitute a rapid yet lucid review of Riemannian geometry and the theory of submanifolds. Most of the results are in Chapters III (AIS's of K-manifolds), IV (AIS's of S-manifolds tangent to ξ) and V (AIS's of S-manifolds normal to ξ). Within these chapters the results are organized into sections so that usually theorems having similar hypotheses are grouped together. Chapter VI (AIS's and Riemannian fibre bundles) is somewhat different in spirit. In it the authors relate the properties of submanifolds of an S-manifold $\bar{M}$ to those of submanifolds of a K-manifold $\bar{N}$ in the situation in which there is a Riemannian fibration π: $\bar{M} \to \bar{N}$ whose fibres are the integral curves of the structure field ξ. The most important example is the standard S^1-fibration π: $S^{2m+1} \to \mathbf{C}P^m$.

General comments. The major strengths of the book under review are its clarity, its organization and its comprehensiveness. Researchers in this topic will find it most useful and should appreciate the considerable care which the authors (and also the publisher) used in its preparation.

A weakness of the book, in my opinion, is that it does not give the reader sufficient information about the general behavior of anti-invariant submanifolds. Almost all the results refer only to the AIS's in some highly restricted class of submanifolds (e.g., minimal submanifolds, submanifolds with parallel second fundamental form, etc.); very few results apply to a "generic" class of AIS's.

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Degree theory, by N. G. Lloyd, Cambridge Tracts in Math., vol. 73, Cambridge Univ. Press, Cambridge, Great Britain, 1978, x + 172 pp., $21.00.

The classical topological degree is a useful tool for investigating the equation $F(x) = p$, where F: $\bar{D} \to \mathbf{R}^n$ is a continuous map of the closure of a bounded open subset D of $\mathbf{R}^n$ and $p \in \mathbf{R}^n$. If $F(x) \neq p$ for $x \in \partial D$ one can associate an integer $\deg(F, D, p)$ to the triple (F, D, p); this integer, called the topological degree of F on D with respect to p, has certain properties–usually referred to as the additivity, homotopy and normalization properties–which axiomatically determine the degree and sometimes make its computation possible.

From the viewpoint of solving $F(x) = p$, the most important fact is that $\deg(F, D, p) \neq 0$ implies existence of $x \in D$ with $F(x) = p$. It is also useful to know that if $G: \overline{D} \to \mathbf{R}^n$ is a continuous map such that $\sup\{\|G(x) - F(x)\|: x \in \partial D\} < \inf\{\|F(x) - p\|: x \in \partial D\}$, then $\deg(F, D, p) = \deg(G, D, p)$; and if $D = D_1 \cup D_2$ where D_1 and D_2 are open sets such that $F(x) \neq p$ for $x \in \partial D_1 \cup \partial D_2 \cup \overline{(D_1 \cap D_2)}$, then $\deg(F, D, p) = \deg(F, D_1, p) + \deg(F, D_1, p)$. Of course if F is C^1, p is a regular value of F, and $J_F(x)$ denotes the determinant of the Jacobian matrix of F at x, it is known that

$$\deg(F, D, p) = \sum_{x \in F^{-1}(p)} \operatorname{sgn}(J_F(x)).$$

The difficulty is to pass from the above formula to a definition for general continuous functions.

The interest of analysts in the topological degree was stimulated when J. Leray and J. Schauder **[11]** showed how a degree could be defined for an important class of maps defined on bounded open subsets D of a Banach space X. It is not hard to see that in an infinite dimensional Banach space one cannot define a useful degree theory for all continuous maps. Leray and Schauder singled out the class of maps of the form $F = I - f$, where I is the identity map and f is a compact map, i.e., f is a map which takes bounded sets to sets with compact closure. Leray and Schauder showed that a degree which preserved all properties of the finite dimensional degree could be defined for such compact perturbations of the identity map. More to the point, many questions about nonlinear equations can be transformed to problems of solving $(I - f)(x) = p$ for x and p in a Banach space and f compact.

In the past ten or fifteen years much effort has been spent on generalizing the Leray-Schauder degree. One now has a degree theory for maps of the form $F = I - f$, where f is a condensing map, for A-proper maps, for maps $F(u) = S(u, u)$, where S is a homeomorphism in one variable and compact in the other, and for certain multivalued maps. We omit definitions and remark only that all these classes are substantial generalizations of the Leray-Schauder class and all arise in applications in analysis. In a somewhat different direction one can define an integer valued degree for certain classes of C^1, proper maps F whose Fréchet derivative at every point is Fredholm of index zero.

Another line of generalization involves the fixed point index of a map f defined on an open subset U of a topological space X, written $i_X(f, U)$. Typically, X might be a finite union of closed, convex sets in a Banach space Y, and the index is an algebraic count of the number of fixed points of f in U with respect to X. If X is a Banach space and f is compact, $\deg(I - f, U, 0) = i_X(f, U)$. Notation here is bad: the term index is also used, in the context of degree theory, for the degree of $F = I - f$ on a small neighborhood of an isolated solution x_0 of $F(x) = p$. Generalizations of the classical fixed point index have involved extending the class of functions and spaces for which such an index can be defined. Broadening the class of spaces is useful for problems in analysis; one may, for example, have function f which are only naturally defined on cones of nonnegative functions.

N. G. Lloyd's book provides a careful and essentially self-contained presentation of some of the topics discussed above. The first three chapters present the finite dimensional degree and some of its applications–the Brouwer fixed point theorem, the Borsuk-Ulam theorem, the Jordan separation theorem and the invariance of domain theorem (a one-to-one continuous map takes open sets to open sets). The fourth chapter describes Leray-Schauder degree. All of this is done in a way which analysts will find congenial; the approach to finite dimensional degree theory is that of E. Heinz **[8]** and involves only the easy version of Sard's theorem and some advanced calculus. Lloyd's proof of the Borsuk-Ulam theorem, and indeed the exposition in the first four chapters, closely follows that in J. T. Schwartz's book **[17]**.

The remaining five chapters of Lloyd's book provide one of the first English language treatments in book form of the generalized degree theories previously mentioned (see, also, **[2]**). The exposition here is sketchier than in the early part of the book, and some proofs use lemmas which are not explicitly stated. Thus the proof of Theorem 8.2.4 uses (without saying so) the fact that a compact one dimensional manifold with boundary is a union of intervals and circles, and the proof of Lemma 9.5.6 uses a result (Theorem 9.3, Chapter 1) from Whyburn's book **[19]**. There are some surprising omissions. Nothing is said about the fixed point index for condensing maps, even though such a fixed point index (at least for condensing maps defined on relatively open subsets U of closed convex sets X in a Banach space Y) could have been easily defined. As a result at least one theorem (Theorem 6.3.4) is proved under unnecessarily restrictive assumptions. The key point is that it is unclear if a strict set contraction $f\colon S \to S$, S a closed convex subset of a Banach space X, has an extension as a strict set contraction to a map $f_1\colon X \to S$; this difficulty, by the way, vitiates Lloyd's proof of Darbo's theorem on p. 103.

The final (and longest) chapter of the book provides some applications of degree theory to analysis, in this case mostly to the question of periodic solution of nonautonomous ordinary differential equations. Some of the applications are pretty, notably a result of Ezeilo and (later) Reissig on existence of periodic solutions for a class of third order ODE's. Also sandwiched into Chapter 9 are sections on degree theory for holomorphic maps and bifurcation theory. The latter section seems something of an afterthought.

One serious criticism must be made of Lloyd's book. There is no historical discussion whatsoever, and the attribution of credit for more recent theorems is often incorrect or incomplete. The reader will not find here that Kronecker had already defined what we would call a degree theory in 1878 (see **[10]**; Hadamard presented an exposition of Kronecker's work in **[7]**). The Poincaré-Bohl theorem appears on p. 25, but the reader will search in vain for any references to Poincaré or Bohl, for any explanation of how the theorem got its name, or for any explanation of why the "Brouwer fixed point theorem" is so named if Poincaré and Bohl had a better theorem earlier.

Moving to more recent times, Chapter 5 discusses axiomatic characterization of degree theory. The crucial point, as was discovered by Führer [**6**], is that the multiplicative formula is not necessary to determine the degree axiomatically in finite dimensions. This result was rediscovered by Amann and Weiss [**1**], who gave the axiomatic treatment presented by Lloyd. However, Führer's work is not cited. Chapter 6 presents a result on uniqueness of the degree for condensing maps which was first proved in the stated generality by this reviewer [**13**], but the article is not cited. An invariance of domain theorem which Lloyd attributes to Webb [**18**] on p. 105 was in fact first proved by this reviewer in his 1969 dissertation and is so acknowledged by Webb [**18**]. There is only one listed reference for Sadov'skiĭ, even though he has worked extensively on the theory of condensing maps (including degree theory). The results of Chapter 8 were obtained (for the harder case of C^1 Fredholm maps) by C. Isnard [**9**], but Isnard's work and other related work is not mentioned. Theorem 9.3.3, for which Lloyd cites his own paper [**12**], is actually a classical result in the degree theory of holomorphic maps. It follows easily from results of J. Cronin [**3**] and is explicitly stated in a 1963 article of J. Schwartz [**16**] and is rederived in an elegant 1973 article by P. Rabinowitz [**15**]. These examples can, unfortunately, be multiplied.

In fairness to Lloyd it must be said that the history of degree theory presents a tangled skein. Disentangling even the recent history would be a difficult job, but the reader can reasonably expect more than is done here.

Should a person who is seriously interested in learning degree theory read Lloyd's book? If he has had no acquaintance with the subject, this is certainly a reasonable introduction–though the previous criticisms must be kept in mind. If he is already familiar with the Leray-Schauder degree, the answer will depend on his interest in the introduction to generalized degree theories to be found in Lloyd's book. The mathematician who reads German may also want to consider a recent book by Eisenack and Fenske [**5**]. The Eisenack-Fenske book touches on the fixed point index and on other results (e.g., the connection between the fixed point index and the Lefschetz fixed point theorem) which are not treated by Lloyd.

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The Minkowski multidimensional problem, by Aleksey Vasil'yevich Pogorelov, translated by V. Oliker, with an introductory comment by L. Nirenberg, Wiley, New York, Toronto, London, Sydney, 1978, v + 106 pp., $13.75.

The book under review is, to the reviewer's knowledge, the first exposition in English of an important topic in geometry since Busemann's text *Convex surfaces* (Interscience, 1958). It is hoped that this review, as well as Nirenberg's *Introductory commentary* which prefaces the English translation, may help popularize this beautiful subject in the English reading mathematical community.

The Minkowski problem, in its original formulation [1],[1] deals with the determination of a closed, convex hypersurface F in euclidean n-space, in terms of a given, positive valued function $f(\xi)$ ($\xi = (\xi_1, \ldots, \xi_n)$, $\sum_i \xi_i^2 = 1$) defined on the unit hypersphere S^{n-1}, where $f(\xi)$ represents the reciprocal of the Gaussian curvature of F at the point where the outward unit normal is the vector ξ. The function f (which we call the Minkowski data) must necessarily satisfy the exactness condition expressed by the vector equation

$$\int \xi f(\xi)\, d\omega(\xi) = 0, \tag{1}$$

the integration being meant over the sphere S^{n-1}.

This problem was solved originally by Minkowski only in the following, "weak" sense: given the Minkowski data satisfying (1), there exists a closed, convex hypersurface F, unique up to a translation, such that, for any given, closed region $G \subset S^{n-1}$ the integral

$$\int_G f(\xi)\, d\omega(\xi)$$

[1]References in square brackets are in terms of the bibliography at the end of Pogorelov's book.