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BOOK REVIEWS

Anti-invariant submanifolds, by Kentaro Yano and Masahiro Kon, Lecture Notes in Pure and Applied Mathematics, vol. 21, Marcel Dekker, Inc., New York, 1976, vii + 183 pp., $19.75.

General introduction. This article is divided into two parts: Part I quickly recalls the basic concepts of differential geometry, including the notions of differentiable manifolds and Kählerian and Sasakian manifolds, while Part II is the review of the Yano-Kon book. (Some readers, familiar with differential geometry, may wish to skip Part I. For their benefit I have repeated the definitions of Kählerian and Sasakian manifolds at the start of Part II.)

Part I. Review of differential geometry. A *topological n-manifold* is a metrizable topological space M which locally looks like $\mathbf{R}^n$, in the following sense: each point $p \in M$ has a neighborhood U homeomorphic to some open set W in $\mathbf{R}^n$. If $x = (x_1, \ldots, x_n)$: $U \to W \subset \mathbf{R}^n$ is such a homeomorphism, then the pair (U, x) is called a *coordinate chart*. Two such charts (U, x) and (V, y) are *smoothly related* if either (a) $U \cap V = \varnothing$, or (b) $U \cap V \neq \varnothing$ and the maps $x \circ y^{-1}$: $y(U \cap V) \to x(U \cap V)$ and $y \circ x^{-1}$: $x(U \cap V) \to y(U \cap V)$ (defined on the open subsets $y(U \cap V)$ and $x(U \cap V)$ in $\mathbf{R}^n$) are smooth (i.e., of class C^∞). A *differentiable n-manifold* is a topological n-manifold M on which a class F of coordinate charts has been singled out. The class F must satisfy (a) every $p \in M$ is in some chart of F and (b) if (U, x) and (V, y) are charts in F, then they are smoothly related.

If M is a differentiable n-manifold, one can "do calculus on M". For example, one can introduce the notion of differentiable functions on M: a function f: $M \to \mathbf{R}$ is of class C^k if for each chart $(U, x) \in F$ the function $f \circ x^{-1}$: $x(U) \to \mathbf{R}$ is of class C^k on the open set $x(U)$ in $\mathbf{R}^n$. Similarly, by considering compositions of the form $x \circ \gamma$, one can define a notion of differentiable maps γ from (say) an interval in $\mathbf{R}$ to M.

Associated with each point p of a differentiable manifold M is an n-dimensional real vector space, the *tangent space* $T_p M$. The fact that M admits such "linear approximations" is the central feature of the theory of differentiable manifolds; in particular, it explains the constant use of linear and multilinear algebra so characteristic of this theory. The most intuitive description of the tangent space is this: if γ: $(a, b) \to M$ is a smooth curve at p (i.e., a smooth map into M of an interval $a < t < b$ containing $t = 0$ such that $\gamma(0) = p$) and if $(U, x) \in F$ is a chart containing p, then we can associate to γ a vector in $\mathbf{R}^n$, namely $v_\gamma = d((x \circ \gamma)(t))/dt|_{t=0}$; we say that two such curves at p, γ and σ, are equivalent if $v_\gamma = v_\sigma$ and we denote the equivalence class of γ by $[\gamma]$; then an element of T_pM is simply such an equivalence class and the vector space structure on T_pM is that induced by the bijection $[\gamma] \mapsto v_\gamma$ of T_pM with $\mathbf{R}^n$. (This construction does not depend on the particular chart at p that we happen to use.)

Analogously, we define a *complex n-manifold* to be a metrizable topological space M which is covered by a prescribed family F of "holomorphically related complex charts". That is, if $(U, z) \in F$ then U is open in M and $z = (z_1, \ldots, z_n)$ maps U homeomorphically onto an open set in $\mathbf{C}^n$; also if (U, z) and (V, w) are in F and $U \cap V \neq \varnothing$ then the maps $z \circ w^{-1}$: $w(U \cap V) \to z(U \cap V)$ and $w \circ z^{-1}$: $z(U \cap V) \to w(U \cap V)$ are holomorphic. The (analogously defined) tangent spaces are complex vector spaces of (complex) dimension n. Of course, any complex n-manifold is at the same time a (real) manifold of (real) dimension $2n$.

EXAMPLES. (1) Suppose that $f_1, \ldots, f_k$ are smooth functions on an open domain $D \subset \mathbf{R}^{n+k}$. If at each point of D the gradient vectors of these functions are linearly independent, and if there is at least one point of D at which all the f's vanish, then $M = \{p \in D: f_1(p) = \cdots = f_k(p) = 0\}$ is a differentiable n-manifold. (REMARK. According to H. Whitney's celebrated theorem, every differentiable manifold can be represented by such a "concrete" example.)

(2) In like manner, one obtains many examples of complex n-manifolds by considering the zero-set of a system $f_1, \ldots, f_k$ of *holomorphic* functions defined in a domain of $\mathbf{C}^{n+k}$. Because of the maximum modulus theorem, none of these examples is compact.

(3) The most important *compact* complex manifold is the complex projective space $\mathbf{C}P^n$, defined to be the set of complex lines through the origin in $\mathbf{C}^{n+1}$. Alternatively, it can be defined as the orbit space of the natural action of $S^1 = \{z \in \mathbf{C}: |z| = 1\}$ on $(\mathbf{C}^{n+1})^* = \{w \in \mathbf{C}^{n+1}: w \neq 0\}$.

What we have obtained so far is just the framework for differential geometry. (It is, incidentally, also the framework for several other branches of mathematics, notably differential topology.) In order to get a geometric theory we must impose some geometric structure on this framework. In many subbranches of geometry, including that studied by the Yano-Kon book, such structures arise by imposing certain "linear geometric structures" on each tangent space. Here are the main examples we'll encounter.

(1) An inner product on a real vector space V is a symmetric, positive-definite bilinear form on V. A *Riemannian structure* on a manifold M is a (smooth) choice of inner products on its tangent spaces (i.e., a *field* of inner products). Imposing a Riemannian structure makes M into a Riemannian manifold and allows one to introduce geometric notions such as arclength, volume and parallelism.

EXAMPLE. If M is a submanifold of a Euclidean space $\mathbf{R}^N$, then M inherits a natural Riemannian structure, obtained by restricting the usual inner product on $\mathbf{R}^N$ to the tangent spaces of M.

(2) An almost-complex structure on a real vector space V is a linear endomorphism J: $V \to V$ such that $J^2 = -I$, where I is the identity map. If V admits such a structure then its dimension must be even. An *almost-complex manifold* is a (real) manifold with a (smooth) field of such endomorphisms on its tangent spaces.

EXAMPLE. If M is a complex manifold of complex dimension n, so its tangent spaces are complex vector spaces, then M can also be viewed as an almost-complex manifold with J being multiplication by $\sqrt{-1}$. Not every almost-complex structure arises this way; the criterion is that J satisfy certain partial differential equations (Newlander-Nirenberg Theorem).

(3) Following Yano-Kon, we can define a *Kählerian manifold* to be a manifold M with a Riemannian structure g and almost-complex structure J such that (a) on each tangent space, J is an isometry with respect to g, (b) J satisfies the "Newlander-Nirenberg" equations (so M is actually a complex manifold) and (c) the skew-symmetric bilinear form $\omega(v, w) = g(Jv, w)$ satisfies certain differential equations (for those in-the-know: $d\omega = 0$).

EXAMPLES. (a) $M = \mathbf{C}^n$, with the usual (Euclidean) inner product. (b) $M = \mathbf{C}P^n$, with the "Fubini-Study" structure: first, identify $\mathbf{C}P^n$ as the orbit space of the standard action of S^1 on $S^{2n+1} = \{w \in \mathbf{C}^{n+1}: |w| = 1\}$; the isometric action on S^{2n+1} of the unitary group $U(n+1) \subset SO(2n+2)$ commutes with this S^1-action and thus induces an action of $U(n+1)$ on $\mathbf{C}P^n$. Then the Riemannian structure on $\mathbf{C}P^n$ is (essentially) characterized by being invariant under this $U(n+1)$ action. (c) Any complex submanifold M of $\mathbf{C}^{n+k}$ (as described in Example 2 above) or of $\mathbf{C}P^n$ inherits a natural Kählerian structure by restriction to M of the Kählerian structure on the ambient space.

(4) An almost-contact metric structure on a real vector space V is a quadruple (g, φ, ξ, η), where g is an inner product on V, φ is a linear endomorphism of V, ξ is an element of V and φ is a linear functional on V (i.e., an element of the dual space V^*). These objects must satisfy certain conditions: for each $u,v \in V$, $g(\varphi(u), \varphi(v)) = g(u, v) - \eta(u)\eta(v)$, $\varphi^2(v) = -v + \eta(v)$, $\eta(\varphi(v)) = 0$, $\varphi(\xi) = 0$, $\eta(\xi) = 1$, and $\eta(v) = g(v, \xi)$. (Briefly: φ maps V onto the orthogonal complement $\xi^\perp$ of the unit vector ξ and on $\xi^\perp$ acts as an isometric almost-complex structure.) If V admits such a structure, then V is odd-dimensional. A *Sasakian manifold* is a differentiable manifold M with a (smooth) field of almost-contact metric structures on its tangent spaces such that the "structure fields" g, φ, ξ, η satisfy certain differential equations. (For this article it is not important to know the precise form of these equations.)

REMARKS. (a) Almost-contact metric structures arise naturally in Hamiltonian mechanics. (b) There is an important example of a Sasakian manifold described in Part II.

Whenever one has a vector space on which a certain linear object is prescribed, it is natural to seek out those subspaces which are particularly closely related to that linear object. For example, in the presence of an inner product, pairs of mutually orthogonal subspaces are interesting.

Likewise, if a geometric structure on a manifold M is described by the presence of a certain linear object on each tangent space, it is natural to seek those submanifolds of M whose tangent spaces are "interesting" subspaces (relative to the linear object) of the tangent spaces of M. The object of the Yano-Kon book is to study the following kinds of submanifolds of Kählerian and Sasakian manifolds.

DEFINITION. If $\overline{M}$ is a Kählerian manifold with almost-complex structure J, then a submanifold M in $\overline{M}$ is an *anti-invariant submanifold* if for each $p \in M$, $J(T_pM) \subset (T_pM)^\perp$ ("$\perp$" denotes orthogonal complement in $T_p\overline{M}$); likewise, if $\overline{M}$ is a Sasakian manifold with structure field φ, then $M \subset \overline{M}$ is *anti-invariant* if $\varphi(T_pM) \subset (T_pM)^\perp$.

Submanifold theory has a rather forbidding terminology. I'll end Part I by informally describing some of the words from that theory which the Yano-Kon book frequently uses.

a. *Sectional curvature* is the generalization (to any Riemannian manifold) of the classical notion of the Gaussian curvature of a surface in $\mathbf{R}^3$. All you need know for this book review is that if M has constant sectional curvature c then M is locally isometric to a domain in one of the following "models": Euclidean space ($c = 0$), spherical space ($c > 0$), or hyperbolic (i.e., Lobachevski) space ($c < 0$). (If $c = 0$ one also says that M is *flat*.) The analogous notion in a Kählerian manifold is *constant holomorphic sectional curvature* c; the corresponding "model spaces" are $\mathbf{C}^n$ (if $c = 0$), $\mathbf{C}P^n$ (if $c > 0$), and the unit disc in $\mathbf{C}^n$ with the "Bergman kernel metric" (if $c < 0$).

B. In Riemannian geometry the "straightest curves" (i.e., the analogs to the straight lines of Euclidean geometry) are called *geodesics*. A submanifold whose geodesics are also "straight" (i.e., are geodesics) in the geometry of the ambient manifold is said to be *totally geodesic*. (Examples: linear subspaces in $\mathbf{R}^N$; equators in S^N.)

C. To each point p in a submanifold M of a Riemannian manifold $\overline{M}$ we can associate a bilinear map B: $T_p(M) \times T_p(M) \to (T_p(M))^{\perp}$ called the *second fundamental form of M in $\overline{M}$*. (The precise construction of B is not important here.) The inner product on $T_p(\overline{M})$ determines an inner product on the space of such bilinear maps and thus a *norm* $\|B\|$ of B. (It is worth noting that $\|B\| = 0$ on M if and only if M is totally geodesic in $\overline{M}$.) If trace$(B) = 0$ we say that M is a *minimal* submanifold of $\overline{M}$. Finally, one can make sense of the notion that B is "constant" (or, as it is more commonly said, *parallel*) on M. Good examples are the linear and spherical subspaces of $\mathbf{R}^N$, and the spherical subspaces of S^N.

Part II. Review of the Yano-Kon book.

Introduction. The book under review is a carefully organized detailed survey of what is known about anti-invariant submanifolds of Kählerian and Sasakian manifolds. The study of such objects began only about eight years ago, but already quite a few geometers, for example B. Y. Chen, C. S. Houh, G. D. Ludden, K. Oguie, M. Okumura and the authors, have published results in this area.

Notation. I shall write AIS for anti-invariant submanifold, K-manifold for Kählerian manifold and S-manifold for Sasakian manifold.

DEFINITIONS. First, recall that a K-manifold is a Riemannian manifold $\overline{M}$ with a (smooth) field of linear isometries J: $T_p\overline{M} \to T_p\overline{M}$ $(p \in \overline{M})$ of its tangent spaces. This tensor field J (the "almost-complex structure") must satisfy $J^2 = -I$ (hence $\dim(\overline{M})$ is even) as well as certain differential equations. Similarly, an S-manifold is an odd-dimensional Riemannian manifold $\overline{M}$ with a unit tangent vector field ξ (the "structure vector field") and a field of linear operators φ: $T_p\overline{M} \to T_p\overline{M}$ such that, for each p in $\overline{M}$, $\varphi(\xi_p) = 0$, φ maps $\xi_p^{\perp}$, the orthogonal complement of ξ_p in $T_p\overline{M}$, isometrically onto itself, $\varphi^2 = -I$ on $\xi_p^{\perp}$, and the fields φ and ξ satisfy certain differential equations. Next, a submanifold M of a K-manifold $\overline{M}$ is said to be an AIS if for each p in M, $J(T_pM)$ is orthogonal to T_pM in $T_p\overline{M}$. Likewise a submanifold M of an S-manifold $\overline{M}$ is said to be an AIS if for each p in M, $\varphi(T_pM)$ is orthogonal to T_pM.

REMARK. The codimension $\dim(\overline{M}) - \dim(M)$ of an AIS must be at least $\dim(M)$ if $\overline{M}$ is Kählerian and at least $\dim(M) - 1$ if $\overline{M}$ is Sasakian. Also, an AIS in a K-manifold is often called a totally real submanifold.

EXAMPLES. (A) The usual embedding of $\mathbf{R}^m$ into $\mathbf{C}^n$.

(B) The usual embedding of $\mathbf{R}P^m$ into $\mathbf{C}P^n$.

(C) $M = S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{n+1})$, where $S^1(r) = \{z \in \mathbf{C}: |z| = r\}$ and $r_1^2 + \cdots + r_{n+1}^2 = 1$, and $\overline{M} = S^{2n+1}$, the unit sphere in $\mathbf{C}^{n+1}$. The embedding is simply by set-inclusion.

(D) $M = S^1 \times S^1 \times \cdots \times S^1$ (n factors) and $\overline{M} = S^{2n+1}$. The embedding is given by the formula $(z_1, z_2, \ldots, z_n) \to (z_1, \ldots, z_n, (z_1 \cdot z_2 \cdot \ldots \cdot z_n)^{-1})/\sqrt{n+1}$. The Riemannian structure on M is that induced by this embedding.

In Examples (A) and (B) the Kählerian structures on $\mathbf{C}^n$ and $\mathbf{C}P^n$ (respectively) are the standard ones. In Examples (C) and (D) the Sasakian structure on S^{2n+1} is derived from the almost-complex structure on $\mathbf{C}^{n+1}$: $\xi_p = J(p)$ and $\varphi = \pi \circ J$, where $\pi: T_p(\mathbf{C}^{n+1}) \to T_p(S^{2n+1})$ is the orthogonal projection.

We may obtain more examples by observing that every submanifold of an AIS is also an AIS. Moreover, any submanifold of an S-manifold which is everywhere orthogonal to the structure vector field is an AIS.

The book. The authors were able to compile, in a moderate amount of space, a very large number of results. (For example, Chapter III is 36 pages long and contains 13 lemmas, 12 propositions, 22 theorems and 10 corollaries.) Because the book is a compendium, its impact is perhaps more diffuse than that of a work which focuses sharply on some single main goal (such as a structure theorem). Of course some of the results in the book are more interesting than others, but no one theorem overshadows the rest. I'll list two results, typical of those presented by the authors, just to give the flavor of the theory.

(Corollary 6.2 of Chapter III). Let M be an AIS of a K-manifold $\overline{M}$ of constant holomorphic sectional curvature (e.g., $\overline{M}$ could be $\mathbf{C}^n$ or $\mathbf{C}P^n$). If M is minimal, of constant sectional curvature, and with parallel second fundamental form, then either M is totally geodesic or flat.

(Theorem 5.3 of Chapter IV). Let M be an $(n + 1)$-dimensional compact AIS in S^{2m+1} tangent to ξ. If the second fundamental form is parallel and the square of its norm has constant value $(5n^2 - n)/(2n - 1)$, and if M is minimal then M is $S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3})$ in some $S^5 \subset S^{2m+1}$. (See Example (C) of this review.)

I found those theorems which, like the second of these examples, characterize one of the "standard" AIS's to be the most interesting.

The authors use only standard differential-geometric tools: the Gauss curvature equations, the Codazzi equations, the Laplacian of the second fundamental form, integral formulas and so on; usually they assume that the ambient manifold $\overline{M}$ is a Kählerian or Sasakian "space form" (the analogue of a Riemannian manifold of constant sectional curvature in Riemannian geometry) although a few results, which give sufficient conditions for M to be conformally flat, require instead that $\bar{M}$ have vanishing Bochner tensor. The well-organized proofs and calculations are cleanly presented in a straightforward easy-to-follow manner and, despite the many indices, are nearly always free of errors, even typographical. (Two exceptions: The proof of Theorem 4.1 of Chapter III–and its analogues in later chapters–does not make it clear whether the distribution L lives in $\bar{M}^m(4)$ or in the frame bundle of that manifold; in Chapter IV, Example 8.1 appears to contradict Proposition 10.2, but including the hypothesis $c \geqslant 1$ fixes it up.)

The organization of the book is straightforward and enhances its role as a reference work. Chapters I and II constitute a rapid yet lucid review of Riemannian geometry and the theory of submanifolds. Most of the results are in Chapters III (AIS's of K-manifolds), IV (AIS's of S-manifolds tangent to ξ) and V (AIS's of S-manifolds normal to ξ). Within these chapters the results are organized into sections so that usually theorems having similar hypotheses are grouped together. Chapter VI (AIS's and Riemannian fibre bundles) is somewhat different in spirit. In it the authors relate the properties of submanifolds of an S-manifold $\bar{M}$ to those of submanifolds of a K-manifold $\bar{N}$ in the situation in which there is a Riemannian fibration $\pi: \bar{M} \to \bar{N}$ whose fibres are the integral curves of the structure field ξ. The most important example is the standard S^1-fibration $\pi: S^{2m+1} \to \mathbf{C}P^m$.

General comments. The major strengths of the book under review are its clarity, its organization and its comprehensiveness. Researchers in this topic will find it most useful and should appreciate the considerable care which the authors (and also the publisher) used in its preparation.

A weakness of the book, in my opinion, is that it does not give the reader sufficient information about the general behavior of anti-invariant submanifolds. Almost all the results refer only to the AIS's in some highly restricted class of submanifolds (e.g., minimal submanifolds, submanifolds with parallel second fundamental form, etc.); very few results apply to a "generic" class of AIS's.

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Degree theory, by N. G. Lloyd, Cambridge Tracts in Math., vol. 73, Cambridge Univ. Press, Cambridge, Great Britain, 1978, x + 172 pp., $21.00.

The classical topological degree is a useful tool for investigating the equation $F(x) = p$, where $F: \bar{D} \to \mathbf{R}^n$ is a continuous map of the closure of a bounded open subset D of $\mathbf{R}^n$ and $p \in \mathbf{R}^n$. If $F(x) \neq p$ for $x \in \partial D$ one can associate an integer $\deg(F, D, p)$ to the triple (F, D, p); this integer, called the topological degree of F on D with respect to p, has certain properties–usually referred to as the additivity, homotopy and normalization properties–which axiomatically determine the degree and sometimes make its computation possible.