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A HIGHER DIMENSION GENERALIZATION OF THE SINE-GORDON EQUATION AND ITS BÄCKLUND TRANSFORMATION

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The classical Bäcklund theorem ([1], [4], [5]) studies the transformation of hyperbolic (i.e. constant negative curvature) surfaces in R^3 by realizing them as focal surfaces of pseudo-spherical line congruences. The integrability theorem says that one can construct a family of new hyperbolic surfaces in R^3 from a given one. Bianchi showed how to construct algebraically another family of hyperbolic surfaces from this family.

It is well known that there is a correspondence betwen solutions of the Sine-Gordon equation

SGE $$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi$$

and hyperbolic surfaces in R^3 ([1], [4], [5]). Therefore Bäcklund's theorem provides a method for generating new solutions of SGE from a given one, and Bianchi's permutability theorem [5] enables one to construct more solutions by an algebraic formula. This technique has recently received much attention in the studies of soliton solutions of SGE [2] and has been used successfully in the study of solitons of other nonlinear equations of evolution in one space dimension. But generalizations to more space variables has been less successful.

A natural generalization would be to find a transformation theory for hyperbolic (i.e. constant negative sectional curvature) submanifolds in Euclidean space. É. Cartan [3] showed that hyperbolic n-manifolds locally immerse in R^{2n-1}, but not in R^{2n-2}. Moreover, [3] he proved the existence of "line of curvature coordinates", in which all components of the second fundamental form are diagonalized. J. D. Moore [6] improved this result and we have:

THEOREM 1 (É. CARTAN). *Suppose M is a hyperbolic n-submanifold of R^{2n-1}. Then locally M can be parametrized by its lines of curvature so that*

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$$\mathrm{I} = \sum_{i=1}^{n} (a_i)^2 du_i^2,$$

$$\mathrm{II} = \sum_{i=1}^{n} \sum_{m=n+1}^{2n-1} b_{im}(a_i)^2 du_i^2 e_m,$$

where $\Sigma_{i=1}^n (a_i)^2 = 1$ and $e_{n+1}, \ldots, e_{2n-1}$ is an orthonormal local frame field for the normal bundle of M. In particular, the normal bundle is flat. Moreover, $\Sigma_{i=1}^n \partial/\partial u_i$ is the unique unit asymptotic vector in the first orthant.

We call such coordinates "generalized Tchebyshef coordinates".

DEFINITION 1. Let E_1 and E_2 be two k-planes in a $2k$-dimensional inner product space $(V, \langle\ \rangle)$ and P: $V \longrightarrow E_1$ the orthogonal projection. Define a symmetric bilinear form on E_2 by $(v_1, v_2) = \langle P(v_1), P(v_2)\rangle$. The k angles between E_1 and E_2 are defined to be $\theta_1, \ldots, \theta_k$, where $\cos^2\theta_1, \ldots, \cos^2\theta_k$ are the k eigenvalues for the selfadjoint operator A: $E_2 \longrightarrow E_2$ such that $(v_1, v_2) = \langle A v_1, v_2\rangle$.

DEFINITION 2. A line congruence between two n-submanifolds M and M^* in R^{2n-1} is a diffeomorphism l: $M \longrightarrow M^*$ such that for $P \in M$ the line joining P and $P^* = l(P)$ is a common tangent line for M and M^*.

For a line congruence l: $M \longrightarrow M^*$ between two n-submanifolds in R^{2n-1}, the normal planes ν_P and $\nu^*_{P^*}$ at corresponding points P and P^* are of dimension $n-1$ and are perpendicular to $\overrightarrow{PP^*}$. Therefore they lie in a common $2n-2$ dimensional inner product space, so there are $n-1$ angles between them.

DEFINITION 3. A line congruence l: $M \longrightarrow M^*$ between two n-submanifolds in R^{2n-1} is called pseudo-spherical (p.s.) if

(1) The distance (between P and P^*) is a constant r, independent of P.

(2) The $n-1$ angles between ν_P and $\nu^*_{P^*}$ are the same and equal to a constant θ, independent of P.

(3) The normal bundles ν and ν^* are flat.

(4) The bundle map Γ: $\nu \longrightarrow \nu^*$ given by the orthogonal projection commutes with the normal connections.

Then we have the following generalization of Bäcklund's theorem.

THEOREM 2. *Suppose there is a p.s. congruence* l: $M \longrightarrow M^*$ *of n-submanifolds in* R^{2n-1} *with distance r and angle θ. Then both M and M^* have constant sectional curvature* $-((\sin\theta)/r)^2$.

THEOREM 3. *Suppose M is a hyperbolic n-submanifold in* R^{2n-1} *with sectional curvature* $K = -((\sin\theta)/r)^2$, *where r and θ are constants. Let* $v_1^0, \ldots, v_n^0$ *be the orthonormal base at* P_0 *consisting of principal curvature vectors, and* $v_0 = \Sigma_{i=1}^n c_i v_i^0$ *a unit vector with* $c_i \neq 0$ *for* $1 \leqslant i \leqslant n$. *Then there exists a local n-submanifold* M^* *of* R^{2n-1} *and a p.s. congruence* l: $M \longrightarrow M^*$ *such that if* $P_0^* = l(P_0)$ *we have* $\overrightarrow{P_0P_0^*} = rv_0$ *and θ is the angle between the normal planes at* P_0 *and* P_0^*.

The above results are joint work of both authors; the following results were obtained by the second author.

THEOREM 4. *Suppose* l: $M \longrightarrow M^*$ *is a p.s. congruence of hyperbolic* n-*submanifolds in* R^{2n-1}. *Then the generalized Tchebyshef coordinates (hence lines of curvature and asymptotic curves) correspond under* l.

Bianchi's permutability theorem generalizes to

THEOREM 5. *Let* l_1: $M_0^n \longrightarrow M_1^n$, l_2: $M_0^n \longrightarrow M_2^n$ *be two p.s. congruences in* R^{2n-1} *with angles* θ_1, θ_2 *respectively. If* $\theta_1 \neq \theta_2$, *then there exists a unique hyperbolic* n-*submanifold* M_3 *of* R^{2n-1} *and p.s. congruences* $\bar{l}_1$: $M_1 \longrightarrow M_3$, $\bar{l}_2$: $M_2 \longrightarrow M_3$ *with angles* θ_2, θ_1 *respectively.*

For the analytic part of this theory, one needs to find the appropriate partial differential equations.

In what follows M will be a hyperbolic n-submanifold (with curvature -1) in R^{2n-1}, and $(u_1, \ldots, u_n)$ etc. as in Theorem 1. Associate to M a map $A = (a_{ij})$: $R^n \longrightarrow O(n)$ defined by

$$a_{1j} = a_j, \qquad 1 \leqslant j \leqslant n,$$

$$a_{ij} = b_j^{n+i-1} a_j, \qquad 1 \leqslant j \leqslant n, 2 \leqslant i \leqslant n.$$

Then A satisfies the following second order system given by the Gauss and Codazzi equations:

$$\frac{\partial}{\partial u_j}\left(\frac{1}{a_{1j}}\frac{\partial a_{1j}}{\partial u_j}\right) + \frac{\partial}{\partial u_i}\left(\frac{1}{a_{1i}}\frac{\partial a_{1j}}{\partial u_i}\right) + \sum_{k \neq i,j} \frac{1}{a_{1k}^2}\frac{\partial a_{1i}}{\partial u_k}\frac{\partial a_{1j}}{\partial u_k} = a_{1i}a_{1j}, \qquad i \neq j,$$

GSGE
$$\frac{\partial}{\partial u_k}\left(\frac{1}{a_{1j}}\frac{\partial a_{1i}}{\partial u_j}\right) = \frac{1}{a_{1k}a_{1j}}\frac{\partial a_{1i}}{\partial u_k}\frac{\partial a_{1k}}{\partial u_j}, \qquad i, j, k \text{ distinct},$$

$$\frac{\partial a_{jk}}{\partial u_i} = \frac{a_{ji}}{a_{1i}}\frac{\partial a_{1k}}{\partial u_i}, \qquad i, j, k \text{ distinct}.$$

Conversely, the complete integrability of the Gauss and Codazzi equations implies that there exists a hyperbolic n-submanifold of R^{2n-1} for a given solution of GSGE, so there is a correspondence between {hyperbolic n-submanifolds of R^{2n-1}} and {A: $R^n \longrightarrow O(n)$, solutions of GSGE}, and the correspondence is unique up to a left-translation by a constant $O(n-1)$ matrix or a diagonal $O(n)$ matrix. For $n = 2$, we have $A = \begin{pmatrix}\cos\phi & \sin\phi \\ \sin\phi & -\cos\phi\end{pmatrix}$ and GSGE is $\partial^2\phi/\partial u_1^2 - \partial^2\phi/\partial u_2^2 = \sin\phi\cos\phi$, hence GSGE is a generalization of SGE to higher dimensions.

Let l: $M \longrightarrow M^*$ be a p.s. congruence with angle $\theta, e_1, \ldots, e_{2n-1}$ an orthonormal frame field on M such that the induced normal connection with respect to normal frame $e_{n+1}, \ldots, e_{2n-1}$ is zero, and $v_1, \ldots, v_n$ the orthonormal frame field on M consisting of principal curvature vectors. Suppose $e_i = \Sigma_{j=1}^n x_{ij} v_j$, for $1 \leqslant i \leqslant n$, then $X = (x_{ij})$ is the corresponding $O(n)$-map for M^*. Hence one has the following analytic formulations of the geometric theorems:

THEOREM 6. *Let $A = (a_{ij})$: $R^n \longrightarrow O(n)$ be a solution of GSGE. Then the following first order completely integrable system*:

$$BT(\theta) \qquad (dX)X^t + X\Phi X^t = X\delta A^t D - DA\delta X^t$$

gives a new solution for GSGE, where $\Phi = (\phi_{ij})$ is the Levi-Civita connection 1-form on M, in fact $\phi_{ij} = 1/a_{1i}\, \partial a_{1j}/\partial u_i\, du_j - 1/a_{1j}\, \partial a_{1i}/\partial u_j\, du_i$, $\delta =$ diag$(du_1, \ldots, du_n)$, and $D =$ diag$(csc\ \theta, \cot\theta, \ldots, \cot\theta)$.

THEOREM 7. *Suppose A_0 is a solution of GSGE and A_i's are solutions of GSGE obtained from A_0 by solving $BT(\theta_i)$ for $i = 1, 2$. Then a fourth solution A_3 can be obtained by the following algebraic formula*:

$$(*) \qquad A_3A_0^{-1} = (-D_2 + D_1A_2A_1^{-1})(D_1 - D_2A_2A_1^{-1})^{-1}J,$$

where $D_i =$ diag$(\csc\theta_i, \cot\theta_i, \ldots, \cot\theta_i)$, and $J =$ diag$(-1, 1, \ldots, 1)$.

REMARK 1.

$$\begin{pmatrix} D_1 & -D_2 \\ -D_2 & D_1 \end{pmatrix}$$

is an element in $O(n, n)$, a group which acts on $O(n)$ by linear fractional transformation, hence the right-hand side of (*) belongs to $O(n)$.

REMARK 2. For $n = 2$, $BT(\theta)$ is the classical Backlund transformation for SGE:

$$\frac{\partial\alpha}{\partial u_1} + \frac{\partial\phi}{\partial u_2} = \cot\theta \cos\alpha \sin\phi + csc\ \theta \sin\alpha \cos\phi$$

$$\frac{\partial\alpha}{\partial u_2} - \frac{\partial\phi}{\partial u_1} = -\cot\theta \sin\alpha \cos\phi - csc\ \theta \cos\alpha \sin\phi,$$

where

$$A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}, \quad X = \begin{pmatrix} \cos\alpha & \cos\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

The formula (*) above is called Bianchi's superposition formula by physists, and becomes

$$\tan\frac{\phi_3 - \phi_0}{2} = \frac{\cos\theta_2 - \cos\theta_1}{\cos(\theta_2 - \theta_1)^{-1}} \tan\frac{\phi_2 - \phi_1}{2},$$

where

$$A_i = \begin{pmatrix} \cos\phi_i & \sin\phi_i \\ \sin\phi_i & -\cos\phi_i \end{pmatrix}.$$

If A is taken to be the identity everywhere one gets a trivial ("vacuum") solution of GSGE. Applying $BT(\theta)$ with varing initial conditions to this solution gives families of solutions including the one dimensional solitary wave solutions of GSE. Finally, applying the superposition formula (*) consecutively to these families gives further families of solutions which generalize the n-solition solutions of SGE. A fuller discussion of these solutions will appear elsewhere together with a proof of the above theorems.

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