BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 3, May 1979

REMOVABLE SINGULARITIES IN YANG-MILLS FIELDS

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In the last several years, the study of gauge theories in quantum field theory has led to some interesting problems in nonlinear elliptic differential equations. One such problem is the local behavior of Yang-Mills fields (defined below) over Euclidean 4-space. Our main result is a local regularity theorem: A Yang-Mills field with finite energy over a 4-manifold cannot have isolated singularities. Apparent point singularities (including singularities in the bundle) can be removed by a gauge transformation. In particular, a Yang-Mills field for a bundle over R^4 which has finite energy may be extended to a smooth field over a smooth bundle over $R^4 \cup \{\infty\} = S^4$.

I would like to thank R. S. Palais for encouraging me to look at this problem.

The Yang-Mills equations are equations on the Lie algebra valued 2-form Ω, the curvature or field of a connection (or vector potential) for a bundle η over a Riemannian manifold M. For our purposes, η has structure group $G \subset SO(n)$, Lie algebra $\mathfrak{A}$, and fiber $\eta_x = R^n$. Our local estimates assume $M = U \subset R^4$, and the gauge group is the group of $\{g \in C^\infty(U, G)\}$.

A covariant derivative is a first order operator of the form

$$D = d + \Gamma, \qquad \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4).$$

This transforms under an element of the gauge group to

$$g^{-1}Dg = d + g^{-1}dg + g^{-1}\Gamma g.$$

The curvature form is $\Omega = \{\Omega_{ij}\} = \{\partial/\partial x^j\Gamma_i - \partial/\partial x^j\Gamma_j + [\Gamma_i, \Gamma_j]\}$ where Γ_i and Ω_{ij} are skew $k \times k$ matrices in $\mathfrak{A}$. We say D is the covariant derivative of a Yang-Mills field Ω, if Γ is a critical point of the energy

$$E(\Gamma) = \sum_{i,j} \int - tr\, \Omega_{ij}^2\, dx = \int |\Omega|^2\, dx$$

which implies Ω satisfies the Euler-Lagrange equations

$$D^*\Omega = \sum_i \partial/\partial x^i\Omega_{ij} + [\Gamma_i, \Omega_{ij}] = 0.$$

Received by the editors December 4, 1978.
AMS (MOS) subject classifications (1970). Primary 35J60.

Solutions of those Euler-Lagrange equations are invariant under the action of the gauge group, and in an improper choice of gauge may fail to be smooth.

Our first observation is independent of choice of gauge. It results from manipulating $D^*\Omega = 0$ and the Bianchi identities $D\Omega = 0$ to get an equation of the form

$$\Delta_D \Omega + 2[\Omega, \Omega] = 0.$$

Standard Moser type estimates are used to estimate the maximum of the subharmonic function $|\Omega|^2$.

THEOREM A. *If Ω is Yang-Mills, then*

$$\Delta|\Omega|^2 \leqslant 4 \sum_{i,j,k} \operatorname{trace}(\Omega_{ij} \cdot [\Omega_{j,k}, \Omega_{ki}]).$$

There exists $\alpha > 0$, such that if $D(r, p)$, the disk of radius r about p, lies in U and $\int_{D(r,p)} |\Omega|^2\, dx < \alpha$, then

$$|\Omega(p)|^2 \leqslant r^{-4} c \int_{D(r,p)} \Omega^2\, dx.$$

As a corollary, we notice that if zero is a point singularity, then $|\Omega(X)| \leqslant |x|^{-2}\tilde{c}$. Due to conformal invariance, $\tilde{c}$ is not uniform. By the same argument, we may change coordinates and assume $|\Omega(x)| \leqslant |x|^{-2}\epsilon$ for any $\epsilon > 0$.

In order to proceed further, we must be able to make a canonical choice of gauge (or coordinates) in η. We define a *generalized Coulomb gauge* as one in which $d^*\Gamma = \Sigma_i\, \partial/\partial x^i \Gamma_i = 0$ and prove that such a gauge may be chosen if the curvature is small using the implicit function theorem. Our main technical construction in removing point singularities is that of a broken Coulomb gauge around a singularity (at the origin). This gauge is smooth in annuli $A_i = \{x: 2^{-i} \leqslant |x| \leqslant 2^{-i+1}\}$, well defined and Coulomb on the bundle $\eta|S_i$, where $S_i = \{x: |x| = 2^{-i}\}$ and estimated by the curvature terms in A_i. Our basic estimate comes from integrating Γ in this gauge as a test function against the Euler-Lagrange equations:

$$\int (\Gamma \cdot D^*\Omega) = \int D\Gamma \cdot \Omega + \text{boundary terms}.$$

If $\omega = \max_{|x| \leqslant 1} |x|^4 |\Omega(x)|^2$, we get

$$4(1 - K\omega) \int_{|x| \leqslant 1} |\Omega|^2 \leqslant \int_{|x|=1} |\Omega|^2.$$

By expansion of a disk $D(r)$ we get

$$4(1 - K\omega) \int_{|x| \leqslant r} |\Omega|^2 \leqslant r \int_{|x|=r} |\Omega|^2.$$

This inequality integrates to give

$$\int_{|x|\leqslant r} |\Omega|^2 \leqslant r^{4/1-K\omega} \int_{|x|\leqslant 1} |\Omega|^2.$$

This, coupled with Theorem A, bounds $\|\Omega\|_\infty$.

THEOREM B. *If Ω is a Yang-Mills field in a bundle η over a punctured disk $\{x: 0 < |x| \leqslant 1\} \subset R^4$ and $\int |\Omega|^2\, dx < \infty$, then $|\Omega(x)|$ is bounded on $|x| \leqslant 1/2$. There exists a gauge in which η, D and Ω smoothly extend across $x = 0$.*

COROLLARY. *If Ω is a Yang-Mills field in the complement of a disk in R^4 and $\int |\Omega|^2 < \infty$, then $|x|^4 |\Omega(x)|$ is bounded on $|x| \geqslant 2$. The map $x \longrightarrow 1/x$ transforms Ω into a Yang-Mills field over the unit disk with a removable singularity.*

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