more elementary or, at least, less condensed exposition is possible.

Lectures on closed geodesics is an uncompromising, essentially self-contained, exposition of the Hilbert manifold approach to Morse theory on $\bigwedge M$. In addition, Lyusternik's method of subordinated homology classes is developed and used when needed. Chapter 3, whose central theme is the index theorem, includes extensive material on symplectic geometry and the Poincaré map which has not previously appeared in book form. Included in the last chapter is a far ranging report on manifolds of elliptic and hyperbolic type, integrable and Anosov geodesic flows, and manifolds without conjugate points. An appendix gives more elementary proofs of the Lyusternik-Fet and Lyusternik-Schnirelmann theorems. J. Moser and J. Sacks have contributed sections, respectively, on the Birkhoff-Lewis fixed point theorem and Sullivan's theory of minimal models.

So if this is the most up-to-date, most complete exposition available, where is a student to begin? Certainly not here. A more reasonable route into the calculus of variations in the large (periodic geodesic division) would be to start with Smale's review [8], then Milnor's *Morse Theory* [5], followed by Seifert-Threlfall [7], Alber's Uspehi surveys [1] and [2] (the difference between the 1957 and 1970 ones is historically interesting), and then, perhaps in conjunction with a stay in Bonn where Klingenberg gathers an active group of students and coworkers, *Lectures on closed geodesics*.

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Optimal stopping rules, by A. N. Shiryayev, Applications of Math. vol. 8, Springer-Verlag, New York, Heidelberg, Berlin, x + 217 pp., $24.80.

In the general optimal stopping problem one imagines a gambler observing the outcomes of a sequence of games of chance. The fortune of the gambler depends on these outcomes, and after the nth game in the sequence it equals f_n ($n = 1,2, \dots$). The gambler cannot influence the outcomes of the various games, but he can quit playing whenever he wants. If he quits at the time T, which may depend on how well he is doing, his terminal fortune is f_T. The problem of optimal stopping is to determine that stopping time (if one exists) which maximizes the gambler's *expected* terminal fortune $E(f_T)$.

Optimal stopping theory has its roots in the theory of sequential statistical analysis developed by A. Wald [7] and his colleagues during and following World War II. In brief, one collects data having some statistical regularity which is not immediately discernible due to the presence of random fluctuations. An important consideration in experimental design is the amount of data which will allow one to see through the noise to the underlying pattern. In some experimental situations the data become available over a period of time, and it may be advantageous to choose a time to terminate the experiment based on the data which have already been observed. For example, in a clinical trial to compare two antitoxins for snakebite, one might take $2n$ persons who have been bitten, treat half of them with each antitoxin, and conclude that one is better if its success rate is sufficiently higher than the other. Of course, if the two are about equally efficacious, it will require more data than if one is vastly superior. In absence of *a priori* knowledge of which of these situations prevails, it makes sense to allow data collected early in the clinical trial to indicate the amount of data necessary to reach a reasonable decision. The scientist or engineer seeking to terminate his experiment at the most propitious instant to maximize the available information per unit cost is vaguely analogous to the gambler seeking to maximize his expected terminal fortune, and indeed with more specific assumptions, the "real world" problem of this paragraph and the hypothetical problem of the first paragraph become identical.

For purposes of a brief review it is best to leave the "real world" behind and return to our gambler. The following (fortunately pathological) example indicates some of the intriguing problems which form the subject matter of this book. Suppose that a fair coin is tossed repeatedly, and after the nth toss our gambler's fortune is $f_n = n2^n/(n+1)$ or $f_n = 0$ according as all n tosses have been heads or at least one toss has been tails. Some reflection shows that the only stopping times that need be considered are $T_n = \min(T, n)$, $n = 1, 2, \ldots$, where T denotes the time at which the first tail appears. Then $Ef_{T_n} = Ef_n = n2^n/(n+1) \cdot 2^{-n} \uparrow 1$, although $Ef_T = 0$. Moreover, a gambler who has observed only heads on the first n tosses has a *conditional* expected fortune after $n + 1$ tosses of

$$E(f_{n+1}|f_n > 0) = \left(\frac{n+1}{n+2}\right)2^{n+1} \cdot \frac{1}{2} > \left(\frac{n}{n+1}\right)2^n = f_n,$$

so it would be "foolish" to stop without making at least one more observation. This simple example shows that (i) an optimal stopping time may fail to exist, (ii) behaving optimally in the short run can lead to disaster in the long run, and (iii) the limit of a sequence of increasingly better rules can be a very poor rule. Taking $f'_n = 1 - f_n$ shows one can make money eventually although one plays a sequence of unfavorable games. The purpose of the theory of optimal stopping rules is to produce a general framework which allows one to solve important stopping rule problems while revealing those pathological cases which produce the difficulties of the preceding example.

Optimal stopping rules is a substantial revision of the same author's earlier monograph *Statistical sequential analysis* [5]. It contains a chapter on mathematical preliminaries, chapters on optimal stopping for discrete time and continuous time processes, and a final chapter on applications to mathematical statistics.

Optimal stopping theory in discrete time is a fairly complete subject. Questions of the existence and computation of optimal stopping rules seem to be understood qualitatively, although the number of cases in which one can write down the optimal rule with pencil and paper remains small. Shiryayev gives a thorough account of these results. Perhaps the dominant feature of his exposition is his restriction to sequential games having a stationary Markovian structure. He later shows how apparently more general problems can be viewed in this framework, so in the end nothing has been lost. But initially the additional structure tends to obscure ideas which are true very generally. The fundamental concept of this part of the subject is what has come to be called the "optimality principle of dynamic programming," which made its first appearance in the classic papers of Wald [6] and Arrow, Blackwell, and Girshick [1]. In words it states that the gambler should stop with f_n if and only if this fortune is at least as large as the conditional expected fortune given the outcomes of the first n games of a gambler who always plays at least $n + 1$ games and uses an optimal stopping rule thereafter.

The first part of the chapter on continuous time problems contains results analogous to the discrete time theory, but the proofs are much more technical. Here one sees a justification for restricting things to Markovian problems, for which the necessary technical machinery is available. There is an additional feature of the continuous time theory, which has no analogue in discrete time, and that is the possibility of computing nontrivial solutions with pencil and paper. Discrete time stopping problems involve nonlinear integral equations, which often are difficult to solve even by computer. In continuous time these frequently become familiar (partial) differential equations, and the nonlinearity appears as a free boundary (Stefan) condition. The chapter on continuous time provides a brief introduction to this free boundary theory, which is discussed in slightly more detail in the final chapter on applications. The computational possibilities in continuous time seem the most interesting of the remaining open problems of optimal stopping theory.

The book on the whole is very well written. I have only one reservation (in addition to that previously mentioned concerning the early restriction to Markovian problems). The author has chosen to write a book containing a large number of theorems which are applied to a fairly small number of concrete problems, and most of these appear in the final chapter. This has the effect of making the subject seem less vital and more complete than it is. Examples of applications that I would have liked to see discussed are the recent elegant paper of R. Berk [2] and something from the series of profound papers of H. Chernoff on optimal stopping problems for Brownian motion (cf. [3]). (In making this criticism, I should also express my philosophical position that concrete problems are the life blood of any mathematical subject, and that probability theorists should attempt to preserve the interaction between concrete and abstract which makes their subject both useful and fascinating.)

The book seems to have been written carefully. I did observe the following two errors. In proving that $P_x\{\tau_\varepsilon < \infty\} = 1$ on p. 46, the author seems to have ignored the possibility that $\limsup g(x_n) = \limsup v(x_n) = -\infty$, in which case it does not follow that $g(x_n) > v(x_n) - \varepsilon$ for some n. (A correct version of this argument is given on [**4**, p. 79].) On p. 91, it may be that $M_x^* g(x_{\tau*})$ is undefined–see [**4**, p. 112]. I was also slightly puzzled by the assumptions underlying the example on p. 99. I believe that Shiryayev has tacitly assumed that the random variable ξ is bounded, so that the assumption $|g(x)| \leqslant G < \infty$ on p. 94 and its consequences can be invoked. This assumption is much stronger than necessary, and it can easily be weakened. It is interesting that removing it completely has never been accomplished as an application of any general theorem but through a *tour de force* invented for that purpose by D. Burdick [**4**, p. 57].

The translation is generally good. It does contain some unconventional choices of words which should cause no problems. One example is the repeated use of "independent and uniformly distributed" to mean "independent and identically distributed."

In summary, the book gives an elegant exposition of a fascinating and important albeit specialized subject. It may be viewed as an introduction to dynamic programming ideas in their simplest nontrivial setting, and this should give the book wider readership.

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