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Lectures on closed geodesics, by Wilhelm Klingenberg, Die Grundlehren der Math. Wissenschaften, Band 230, Springer-Verlag, Berlin, Heidelberg, New York, 1978, ix + 227 pp.

Poincaré's 1905 paper *Sur les lignes géodésiques des surfaces convexes* [**6**] considered several problems: (A) the existence of any closed geodesics on convex surfaces, (B) the existence of three distinct simple closed geodesics on surfaces of genus zero, (C) the manner in which families of closed geodesics vary as a surface changes analytically, and (D) the stability of closed geodesics. In the succeeding three-quarters of a century, solutions or outlines of solutions of these problems have provided work for many mathematicians, including such famous men as G. D. Birkhoff, L. Lyusternik, and Marston Morse. And the theory seems to have more than its share of pitfalls: the pioneers published some proofs which did not meet the standards for rigor of later generations, and various ideas have distinguished genealogies of error. (Smale, [**8**, p. 691], cites one case.)

An appreciation of some of the elements involved may be gained by looking at problem (A). Let PM be the space of piecewise differentiable maps, c, of the circle into M, a surface diffeomorphic to S^2. The energy function

$$E(c) = \frac{1}{2}\int_0^1 |\dot{c}(t)|^2\, dt$$

has closed geodesics and constant maps as its critical points. Since PM is connected, one cannot find nontrivial closed geodesics merely by minimizing E. But PM is not simply-connected, so there is hope for finding a critical point by first maximizing E on certain homotopically nontrivial curves in PM, then minimizing the result over a fixed homotopy class. This is the essence of the minimax method, and even in this lowest dimension requires considerable work to make rigorous. However, this approach is the heart of the theory.

One recognizes a preliminary division of labor: the study of the functional E, and the study of the topology of PM. These problems were originally attacked by using finite-dimensional approximations to PM, so that E could be treated by ordinary calculus (or "finite-dimensional Morse theory"). Now, with the language of infinite-dimensional manifolds and fibrations available, the exposition appears much cleaner, but the difficulty remains. PM is often replaced by the Hilbert manifold, ΛM, of absolutely continuous maps with square-summable first derivatives. Stepwise deformations are replaced by a flow; for example, along the gradient of E, when that is defined.

With suitable modifications, the process outlined above works for any dimension and the result is the theorem of Lyusternik and Fet (1951): On every compact Riemannian manifold, there exists at least one closed geodesic. Thus the natural generalization of problem (A) has received a satisfactory answer.

Problem (B) was solved by Lyusternik and Schnirelmann in 1929, using the category theory they invented for the purpose. In fact, they showed that there exist three distinct, non-self-intersecting geodesics on such surfaces. (For the benefit of the younger reader: their categories have little to do with functors. They are closely related to the cohomological cup-length of a space, and, in fact, Lyusternik and his school eventually used variations of the latter more often. For one of the more recent discussions of the topological relationships, see Berstein and Ganea [3].)

In higher dimensions, one would hope that some sort of index theory, analogous to Morse theory for functions on a finite-dimensional manifold, might be effective: after all, that theory obtains estimates for the number of critical points in terms of the topology. But at the outset a disturbing property of the space of closed curves complicates matters: the points representing closed, nontrivial geodesics are never isolated critical points of E. For the circle group (with reflections) changes the map, but not the image set. Dividing out by this action leads to spaces with singularities, so a more fruitful approach has been to consider the orbits as critical submanifolds. A further complication is that an m-fold covering of a closed geodesic registers as a critical point of E different from the once traversed (prime) loop. To deal with these difficulties, Lyusternik and his school extracted more information from the cohomology ring of $\bigwedge M$, while Morse generalized and refined finite-dimensional critical point theory. One of Morse's great contributions was the recognition that natural generalizations of index and nullity yielded finite numbers related to classical conjugate points (the Morse Index Theorem). However, one of the reasons an entire book can be devoted to closed geodesics as distinct from geodesic loops are the subtleties involved in extending the index theorem.

By astute application of Bott's formulas for the indices of multiply covered geodesics and a careful examination of the degenerate case, Gromoll and Meyer [4] were able to provide a surprising answer to the question of how many closed geodesics must exist on a manifold: If the sequence of Betti numbers of $\bigwedge M$ is unbounded, there are infinitely many prime closed geodesics on M. In one direction this remarkable result is not a generalization of Lyusternik-Schnirelmann, for spheres are among the few manifolds which do not satisfy the topological hypothesis. Nor does the Gromoll-Meyer theorem say whether these geodesics can be found without self-intersections.

Klingenberg states two theorems which represent the latest word in these directions. The first asserts that any compact Riemannian manifold with finite fundamental group has infinitely many prime closed geodesics. The second states that a compact, simply connected manifold has $2k - 1$ prime close geodesics which are "relatively short". (Here k depends on the homology of the manifold, but is always $\geqslant 2$.) The hope is that "relatively short" can be changed to "without self-intersections", but this has been done so far only for metrically restricted–for instance, $1/4$-pinched manifolds, and with a different estimate. The proofs of both these theorems of Klingenberg use some constructions reminiscent of Poincaré's problem (C). For at a crucial point he perturbs the metric in order to approximate degenerate critical sets by nondegenerate ones, all the while trying to control the way various level sets and multiplicities behave. The other technique involved is a very delicate application of Sullivan's theory of minimal models. It is to be hoped that a more elementary or, at least, less condensed exposition is possible.

Lectures on closed geodesics is an uncompromising, essentially self-contained, exposition of the Hilbert manifold approach to Morse theory on $\bigwedge M$. In addition, Lyusternik's method of subordinated homology classes is developed and used when needed. Chapter 3, whose central theme is the index theorem, includes extensive material on symplectic geometry and the Poincaré map which has not previously appeared in book form. Included in the last chapter is a far ranging report on manifolds of elliptic and hyperbolic type, integrable and Anosov geodesic flows, and manifolds without conjugate points. An appendix gives more elementary proofs of the Lyusternik-Fet and Lyusternik-Schnirelmann theorems. J. Moser and J. Sacks have contributed sections, respectively, on the Birkhoff-Lewis fixed point theorem and Sullivan's theory of minimal models.

So if this is the most up-to-date, most complete exposition available, where is a student to begin? Certainly not here. A more reasonable route into the calculus of variations in the large (periodic geodesic division) would be to start with Smale's review [8], then Milnor's *Morse Theory* [5], followed by Seifert-Threlfall [7], Alber's Uspehi surveys [1] and [2] (the difference between the 1957 and 1970 ones is historically interesting), and then, perhaps in conjunction with a stay in Bonn where Klingenberg gathers an active group of students and coworkers, *Lectures on closed geodesics*.

References

1. S. I. Alber, *On periodicity problems in the calculus of variations in the large*, Uspehi Mat. Nauk **12** (1957), 57–124; English Transl., Amer. Math. Soc. Transl. (2) **14** (1960), 107–172.

2. ______, *The topology of functional manifolds and the calculus of variations in the large*, Uspehi Mat. Nauk **25** (1970), 57–122 = Russian Math. Surveys **25** (1970), 51–177.

3. I. Berstein and T. Ganea, *Homotopical nipotency*, Illinois J. Math. **5** (1961), 99–130.

4. D. Gromoll and W. Meyer, *Periodic geodesics on compact Riemannian manifolds*, J. Differential Geometry **3** (1969), 493–510.

5. J. Milnor, *Morse theory*, Ann. of Math. Studies,No. 51, Princeton Univ. Press, Princeton, N. J., 1963.

6. H. Poincaré, *Sur les lignes géodésiques des surfaces convexes*, Trans. Amer. Math. Soc. **6** (1905), 237–274.

7. H. Seifert and W. Threlfall, *Variationsrechnung im Grossen*, Teubner, Leipzig, 1938.

8. S. Smale, *Global variational analysis: Weierstrass integrals on a Riemannian manifold, by Marston Morse*, Bull. Amer. Math. Soc. **83** (1977), 683–693.

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Optimal stopping rules, by A. N. Shiryayev, Applications of Math. vol. 8, Springer-Verlag, New York, Heidelberg, Berlin, x + 217 pp., $24.80.

In the general optimal stopping problem one imagines a gambler observing the outcomes of a sequence of games of chance. The fortune of the gambler depends on these outcomes, and after the nth game in the sequence it equals f_n ($n = 1,2, \ldots$). The gambler cannot influence the outcomes of the various