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BOOK REVIEWS

Algebraic geometry I: *Complex projective varieties*, by David Mumford, Grundlehren der Math. Wissenschaften, vol. 221, Springer-Verlag, Berlin, Heidelberg, New York, 1976, x + 186 pp., $14.80.

During the last twenty years, the field of algebraic geometry has undergone a thorough revision and strengthening of its foundations with the introduction of schemes and sheaf cohomology. This period of rapid growth created a significant gap between what could be found in textbooks and the most recent research. These new methods are now having an impact on neighboring fields such as number theory, complex manifolds, topology, and even the nonlinear differential equations of mathematical physics. This creates an even greater need for introductory books which should make algebraic geometry more accessible.

David Mumford has already written several books on various aspects of algebraic geometry [6], [7], [8], [9]. Indeed his "little red book" [6], now out of print, initiated a generation of students to the subject when nothing else was available. More recently a number of introductory books have appeared more or less simultaneously in different parts of the world [1], [2], [4], [5], [10].

In my earlier review [3] of the books by Dieudonné and Shafarevich, I suggested some of the challenges facing the author of an introduction to algebraic geometry: how can he present the complicated technical tools of the new methods and still give enough examples and geometrical intuition to convey what the subject is all about? In the book under review, Mumford neatly sidesteps this difficulty by deferring the technicalities of schemes and cohomology to a second promised volume. To quote from his Introduction, "... the present volume, which is the first of several, introduces only complex projective varieties. But, as a consequence, we can study these effectively with topological and analytic techniques without extensive preliminary work on 'foundations'. My goal is precisely to convey some of the classical geometric ideas and to get 'off the ground': in fact, to get to the 27 lines on the cubic–surely one of the gems hidden in the rag-bag of projective geometry."

I will not attempt here to explain what is algebraic geometry, nor even to summarize what is in the present book. Let me rather try to give the flavor of Mumford's approach by describing three specific topics which he treats.

The first is Chow's theorem which states roughly that the only complex analytic subsets of complex projective space are algebraic varieties. More precisely, this means that if X is a closed subset of $\mathbf{P}^n_{\mathbf{C}}$, and if each point $x \in X$ has a small neighborhood U in $\mathbf{P}^n_{\mathbf{C}}$, such that $X \cap U$ is equal to the set of common zeros of some family of holomorphic functions $f_1, \ldots, f_q$ on U, then there exist homogeneous polynomials $F_1, \ldots, F_s$ in the homogeneous coordinates $z_0, \ldots, z_n$ of the projective space, so that X is exactly the set of common zeros of the F_i in all of $\mathbf{P}^n_{\mathbf{C}}$. The hypotheses state that X is a complex analytic subset of $\mathbf{P}^n_{\mathbf{C}}$, which is a purely local property of X. The conclusion is that X is an algebraic variety, a global property of X.

This theorem is really a result in the theory of several complex variables, not algebraic geometry. Its proof uses analytic techniques such as the Weierstrass Preparation Theorem and Riemann's theorem on the extension of a bounded holomorphic function across an analytic set, which are quoted by Mumford as needed. But it serves to clarify the relationship between algebraic geometry and several complex variables, by showing that a large class of complex analytic objects are in fact algebraic, and so can be studied by the methods of algebraic geometry.

The second topic I will discuss also puts algebraic varieties in a larger context. It characterizes nonsingular algebraic subvarieties of $\mathbf{P}^n_{\mathbf{C}}$ as minimal manifolds in the sense of Plateau's problem. Let M be a compact real oriented $2k$-dimensional submanifold of $\mathbf{P}^n_{\mathbf{C}}$. One knows that the homology $H_{2k}(\mathbf{P}^n_{\mathbf{C}})$ is isomorphic to $\mathbf{Z}$, so the fundamental class of M is an integer $d \geqslant 0$, which we call the *degree* of M. On the other hand let L be a k-complex dimensional linear space $\mathbf{P}^k_{\mathbf{C}}$ contained in $\mathbf{P}^n_{\mathbf{C}}$. Now $\mathbf{P}^n_{\mathbf{C}}$ has a natural Riemannian metric on it, so we can speak of the $2k$-volume of any real $2k$-dimensional submanifold. The theorem is that $\mathrm{vol}(M) \geqslant d \cdot \mathrm{vol}(L)$, and equality holds if and only if M is a nonsingular algebraic subvariety of $\mathbf{P}^n_{\mathbf{C}}$. Thus the algebraic varieties are the submanifolds of minimal volume in each homology class.

Mumford includes this result because it is "the most elementary and intuitive result in the repertoire of Kähler differential geometry, whose higher developments dominate the whole transcendental theory of varieties". The proof includes a discussion of Hermitian forms, singular homology, the fundamental class of a submanifold, and De Rham's theorem. It is thus illustrative of the "transcendental methods" in algebraic geometry, where all these techniques are brought to bear on the study of finer properties of complex algebraic varieties.

The third topic I will single out is the 27 lines on the cubic surface. This classic result of projective geometry states that any nonsingular cubic surface in $\mathbf{P}^3_{\mathbf{C}}$, i.e., given as the zeros of a single homogeneous polynomial of degree 3 in the four coordinates z_0, z_1, z_2, z_3, contains exactly 27 lines (copies of $\mathbf{P}^1_{\mathbf{C}}$), and specifies their incidence relations. This fact, which may appear merely to be a combinatorial curiosity, is actually a striking application of the birational geometry of algebraic surfaces. Two algebraic varieties are said to be *birationally equivalent* if they contain open subsets for the Zariski topology which are isomorphic to each other. This notion, which has no analogue in the theory of differentiable or complex manifolds, gives a fine but nontrivial equivalence relation on algebraic varieties, which is immensely valuable in studying the classification of all varieties. The simplest example of such an equivalence is obtained by *blowing up* a point on a surface: remove the point and replace it by a projective line whose points correspond to the tangent directions on the original surface at the original point. This gives another surface birationally equivalent to the first one.

It turns out that a nonsingular cubic surface in $\mathbf{P}^3_{\mathbf{C}}$ is *isomorphic* to the surface obtained from a projective plane $\mathbf{P}^2_{\mathbf{C}}$ by blowing up 6 points $P_1, \ldots, P_6$ in general position. Once this is understood, the 27 lines appear naturally. They are the new lines replacing each P_i (6 of these), the image of lines in $\mathbf{P}^2$ passing through two of the P_i (15 of these), and the image of conics in $\mathbf{P}^2$ passing through all but one of the P_i (6 of these). This story and the background material that goes into it form a marvelous chapter in the theory of algebraic surfaces.

I hope these three examples will give some idea of what this book is like. It is full of ideas ranging over a wide area but always centered around algebraic varieties in complex projective space. It is not a systematic introduction to algebraic geometry. Rather it is a sampler of a number of methods and results proved by whatever techniques of topology, differential geometry, complex analytic geometry, or commutative algebra lie closest at hand, which should stimulate the interest and imagination of any reader.

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Semisimple Lie algebras, by Morikuni Goto and Frank D. Grosshans, Lecture Notes in Pure and Applied Mathematics, No. 38, Marcel Dekker, Inc., New York, 1978, vii + 480 pp., $37.50.

Lie groups pose a problem for both the learner and the teacher (or textbook writer). Topology, analysis, algebra are so intertwined that no expository scheme can do full justice to the subject without becoming encyclopedic. On the other hand, this diversity of aspect, coupled with a wide range of applicability, makes Lie theory especially attractive.

There are now available a substantial number of books dealing with semisimple Lie groups and/or Lie algebras. These differ considerably in scope and emphasis, but most are built around a common core of Lie algebra theory: nondegeneracy of the Killing form, root system and Weyl group, classification (over $\mathbf{C}$ and perhaps over $\mathbf{R}$), automorphism groups, compact real forms, Cartan decomposition of a real form, finite-dimensional representations, Weyl's character and dimension formulas. This theory reached a certain degree of completeness in the 1930s, following the fundamental work