BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 1, January 1979

SELFADJOINT OPERATOR EXTENSIONS SATISFYING THE WEYL COMMUTATION RELATIONS

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ABSTRACT. Motivated by questions concerning uniqueness of unbounded derivations in commutative C^*-algebras, and related problems on singular perturbations, we define two mixed global and infinitesimal versions of the Weyl operator commutation relations (one degree of freedom and infinite multiplicity), a weak one and a strong one. We announce two structure theorems of a geometric nature which characterize the nonselfadjoint symmetric operators entering in the Weyl systems. Proofs are only indicated.

Our starting point is the following variant of the Stone-von Neumann Uniqueness Theorem [12], [4b]. Let (U, V) be a pair of unitary one-parameter groups (always assumed strongly continuous) of operators on a separable Hilbert space $\mathcal{H}$, and suppose that the Weyl commutation relation

$$U(t)V(s) = V(s)U(t)e^{its} \quad (\text{for all } s, t \in \mathbf{R}) \tag{1}$$

holds. Then it is possible to represent the system in the form $SU(t)S^{-1}f(x) = f(x + t)$, $SV(s)S^{-1}f(x) = e^{isx}f(x)$, where S is an isometry of a space $L^2(\mathbf{R}, \mathcal{N})$ of the norm-square integrable functions f, with values in a separable Hilbert space $\mathcal{N}$, onto $\mathcal{H}$; the dimension of $\mathcal{N}$ being equal to the (uniform) multiplicity of the spectrum of U.[2]

Instead of (1) we consider the following *infinitesimal* Weyl relation with symmetric but generally nonselfadjoint generator. Let $\{U(t)\}_{t\in\mathbf{R}}$ be a unitary one-parameter group on $\mathcal{H}$, and let Q be a symmetric operator with dense domain $\mathcal{D}(Q)$ in $\mathcal{H}$. The corresponding relation

$$(U(t)Qf, g) = (U(t)f, Qg) + t(U(t)f, g) \quad \text{for all } f, g \in \mathcal{D}(Q) \tag{2}$$

is here called the infinitesimal Weyl relation for the triple $(U, Q, \mathcal{H})$. It is clearly equivalent to (1) if Q is essentially selfadjoint. But in scattering theory of singular perturbations, and in recent investigations of the author concerning uniqueness of unbounded derivations, the relation (2) for nonselfadjoint Q plays an interesting role. Simple examples show that the operator Q of a given system $(U, Q, \mathcal{H})$ may

Received by the editors August 3, 1978.

AMS (MOS) subject classifications (1970). Primary 47B25, 81A20; Secondary 43A65, 46K99, 47B47.

Key words and phrases. Weyl relations, operator domains, unitary groups.

[1]Research supported in part by NSF Grant MCS 77-02831 and sequel.

[2]We refer to [LP] as a general reference, containing in addition important applications.

0002-9904/79/0000-0016/$02.00

have nonzero defect indices [4a] which can be equal or unequal. If they are equal, it may *or may not* be possible to extend Q to a selfadjoint operator $\tilde{Q}$ so that the one-parameter group $V(s) = e^{is\tilde{Q}}$ is part of a (global) Weyl system (U, V), i.e. such that (1) holds. The triples $(U, Q, \mathcal{H})$ which can be extended to global Weyl systems are called *extendable.*

In view of the negative examples alluded to above, it may be surprising that a certain canonical and *minimal extension* always exists.

THEOREM 1. *Let $(U, Q, \mathcal{H})$ be an infinitesimal Weyl system. Then symmetric extensions Q_1 of Q exist such that*

$$\begin{aligned} &\mathcal{D}(Q_1) \text{ is invariant under } U(t) \text{ for all } t \in \mathbf{R}, \text{ and} \\ &Q_1 U(t) f = U(t) Q_1 f - t U(t) f \text{ for all } f \in \mathcal{D}(Q_1). \end{aligned} \tag{3}$$

There is a unique smallest symmetric extension Q_m satisfying (3), *i.e. $Q_m \subseteqq Q_1$ for all symmetric extensions Q_1 satisfying* (3). *The operator closure $\bar{Q}_m$ satisfies* (3) *as well and is the unique smallest symmetric and closed extension of Q satisfying* (3).

Here is an easy

COROLLARY 2. *If the system $(U, Q, \mathcal{H})$ is extendable to a global Weyl system (U, V) with $V(s) = e^{is\tilde{Q}}$ then $Q_m \subseteqq \tilde{Q}$.*

DEFINITION. *We say that the system $(U, Q, \mathcal{H})$ has U-indices (p, q) if the minimal operator Q_m has defect indices (p, q).* It can readily be shown that the U-indices are equal whenever Q has equal defect indices, but in general the indices of Q may be infinite while those of Q_m are finite. Theorem 4 below is a converse to Corollary 2.

THEOREM 3. *Let $(U, Q, \mathcal{H})$ be an infinitesimal Weyl system with at least one finite U-index. Then there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a global Weyl system $(\mathcal{U}, \mathcal{V})$ on $\mathcal{H}$ such that $\mathcal{U}$ reduces to U on $\mathcal{H}$, $\mathcal{H}$ is semi-invariant* [9] *for $\mathcal{V}$, and Q is contained in the infinitesimal generator for the contraction semigroup $s \longrightarrow P_{\mathcal{H}} \mathcal{V}(s)|_{\mathcal{H}}$.*

A structure theory for the associated contraction semigroups, due to P. Muhly, will appear in a joint article with the author [2b].

The next result concerns the relation (3) for extendable systems. Hence it is stated for operators in L^2-spaces, and $U(t)$ is translation. Vanishing conditions of the Fourier transform on a set of measure zero clearly give rise to systems (3). The theorem is a partial converse to this statement.

THEOREM 4. *Let Q_1 be a symmetric and closed operator in $L^2(\mathbf{R})$ which is contained in the multiplication operator $\tilde{Q}h(x) = xh(x)$. Assume that $\mathcal{D}(Q_1)$ is translation invariant, and that $\mathcal{D}(\tilde{Q}Q_1)$ is a core for Q_1. Then* (3) *holds and* *the closed set* $\Lambda = \{\lambda \in \mathbf{R} \mid \hat{h}(\lambda) = 0 \text{ for } \forall h \in \mathcal{D}(Q_1)\}$ *is of zero measure. Moreover* $\mathcal{D}(Q_1) = \{h \in \mathcal{D}(\widetilde{Q}) \mid \hat{h}(\lambda) = 0 \text{ for } \forall \lambda \in \Lambda\}$.

The core condition can be omitted in the following cases: (1) Λ *is a Cantor set, or* (2) Q_1 *has finite defect. In general it can be slightly weakened, but not omitted. In case* (2) *the result generalizes to arbitrary multiplicity.*

The proofs of results 1 through 3 involve general operator theory [1], including theorems of Phillips [5] and Naĭmark [3], while the proof of Theorem 4 is based on a function theoretic approach to the extension theory of [4a] made possible by the extendability assumption and the Stone-von Neumann Theorem. (The function theory is based on Wiener-Tauberian consideration.)

Generalizations to group representations [2a], [6], [7], [8], [11] and field theory [10] would be of interest. The possibility of removing the finiteness assumption in Theorem 3 is related via the proof to a well-known conjecture of Phillips [5]. We conjecture the conclusion of Theorem 4, also without the finiteness assumption on the U-index, but it seems hard to settle either of the conjectures. It appears difficult in the general case, for two different extensions to establish the existence of a wave operator similarity which commutes with U.

However the extendability question has an answer in full generality, i.e. no restriction on the indices. The proof uses arguments that naturally extend [4a].

THEOREM 5. *Let* $(U, Q, \mathcal{H})$ *be an infinitesimal Weyl system, and let* $P_\pm$ *denote the orthogonal projections onto the respective defect spaces* $\mathcal{D}_\pm$ *for* Q_m. *The system is extendable if and only if there exists a partial isometry* S *of* $\mathcal{D}_+$ *onto* $\mathcal{D}_-$ *such that* $P_-(2[S, U(t)] - it(I+S)U(t)(I+S))P_+ = 0$ *for all* $t \in \mathbf{R}$. *Here* $[\cdot, \cdot]$ *denotes the commutator bracket.*

If it is assumed in addition that the spectral measure dE_λ *of* U *is absolutely continuous, then extendability is equivalent to the validity of the following identity*

$$P_-(2[S, E_\lambda] - (I+S)D_\lambda(I+S))P_+ = 0 \qquad (\lambda \in \mathbf{R})$$

for some partial isometry S *of* $\mathcal{D}_+$ *onto* $\mathcal{D}_-$. *Here* D_λ *denotes the Radon-Nikodym derivative of* dE_λ.

We finally point out that Theorem 5 has a complete generalization to the case when the W^* algebra generated by the spectral projections of U is replaced by an arbitrary noncommutative W^*-algebra, and when the map $U(t) \longrightarrow it\, U(t)$ is replaced by a spatial derivation which is implemented by a symmetric nonselfadjoint operator. This answers a question raised in a recent article of the author.

The author has benefitted greatly from discussions with Professors P. Chernoff, W. Helton, G. Johnson, R. T. Moore, P. Muhly, and R. S. Phillips. I am pleased to record my gratitude.

REFERENCES

1. C. Foiaş and L. Gehér, *Über die Weylsche Vertauschungsrelationen*, Acta Sci. Math. 24 (1963), 97–102.

2. P. E. T. Jørgensen (a) (with R. T. Moore), *Commutation relations for operators, semigroups, and resolvents in mathematical physics and group representations* (preprint); (b) (with P. S. Muhly), *Selfadjoint extensions satisfying the Weyl operator commutation relations* (in preparation).

3. M. A. Naĭmark, *On commuting unitary operators in spaces with indefinite metric*, Acta Sci. Math. 24 (1963), 177–189.

4. J. von Neumann, (a) *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. 102 (1929–30) 49–131; (b) *Die Eindentigkeit der Schrödingerschen Operatoren*, Math. Ann. 104 (1931), 570–578.

5. R. S. Phillips, *On dissipative operators*, Lecture Series in Differential Equations, vol. II, A. K. Aziz (Ed.), Van Nostrand Mathematical Studies no. 19, Van Nostrand, New York, 1969.

6. N. S. Poulsen, *On C^∞ vectors and interwining bilinear forms for representations of Lie groups*, J. Functional Analysis, (1972), 87–120.

7. R. T. Powers, *On selfadjoint algebras of unbounded operators*, Trans. Amer. Math. Soc. 187 (1974), 261–293.

8. R. R. Rao, *Unitary representations defined by boundary conditions—the case of* $sl(2, R)$, Acta Math. 139 (1977), 185–216.

9. D. Sarason, *On spectral sets having connected complement*, Acta Sci. Math. 26 (1965), 289–299.

10. I. E. Segal, *Nonlinear functions of weak processes*. II, J. Functional Analysis 6 (1970), 29–75.

11. I. M. Singer, *Lie algebras of unbounded operators*, Thesis, Univ. of Chicago, 1950.

12. M. H. Stone, *Linear transformations in Hilbert space*. III. Operational methods and group theory, Proc. Nat. Acad. Sci. U.S.A. 16 (1930), 172–175.

[LP] P. D. Lax and R. S. Phillips, *Scattering theory*, Academic Press, New York, 1967.

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