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*Induced representations and Banach *-algebraic bundles*, by J. M. G. Fell, with an appendix due to A. Douady and L. Dal Soglio-Hérault, Lecture Notes in Math., vol. 582, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 349 pp., $13.70.

Let G be a finite group whose representation theory one wishes to understand. A natural strategy is to consider a subgroup H of G, whose representation theory is presumably simpler, and try to use representations of H to construct representations of G. A method for making such a construction, that of induced representations, was introduced by Frobenius in 1898 [8], and has played a central role in representation theory ever since.

If R is a ring whose representation theory (= module theory) one wishes to understand, then one can use a similar strategy, by considering a subring, S, of R and trying to use S-modules to construct R-modules. General constructions for doing this have become well known as "change-of-ring" operations. Specifically, if M is a left S-module, and if one views R as a right S-module, then one can form $R \otimes_S M$, which is a left R-module. It was not until 1955 that it was pointed out, by D. G. Higman [13], that by using the group algebra of a group, Frobenius' definition of induced representations can be viewed as a special case of this change of ring operation. (Probably the reason this was not noticed earlier is that tensor products in this noncommutative setting had not been clearly formulated until a short time before.) Of course in the group case one goes on to exploit the richer information available essentailly from the fact that the elements of the group form a basis for the group algebra.

Frobenius' definition of induced representations was generalized to unitary representations of locally compact groups by Mackey [18], [19], [20], following the formulation of induced representations in some special cases by Wigner (for the Lorentz group) and others. In the study of the unitary representation theory of non-Abelian noncompact groups (such as the Lorentz group), where the unitary representations tend to be infinite dimensional, induced representations play an even more central role than for finite groups–in fact often a preeminent role.

During the last decade it has become important to consider representations of more general systems, usually consisting of a group and an algebra together with additional pieces of structure. In the simplest such type of system one has a locally compact group, G, a Banach algebra with involution, A, and an action of G as automorphisms of A (given by a homomorphism, p, of G into the automorphism group of A, continuous for the strong operator topology). Such systems have come to be called "covariant systems", and were first studied in a systematic way by the physicists Doplicher, Kastler and Robinson [6], who were interested in the situation in which A is the C^*-algebra of observables of a physical system and G is a group of symmetries of that system (such as the Lorentz group). The representations one needs to study, called covariant representations, consist of a pair (U, π) in which U is a strongly continuous unitary representation of G and π is a nondegenerate $*$-representation of A on the same Hilbert space as U, satisfying the covariance condition

$$U(x)\pi(a)U(x)^* = \pi(p(x)a)$$

for all $x \in G$ and $a \in A$.

In more complicated types of systems one also has other pieces of structure, such as cocycles, or homomorphisms into double-centralizer algebras, which must be taken into account. Thorough references to papers discussing these various types of systems are given in the notes under review.

In most studies of these various kinds of systems, appropriate kinds of induced representations again provide the central tool for developing the representation theory. The tactics consist of showing that if H is a subgroup of G, then the restriction to H of the various pieces of structure gives a system of similar type, and then of showing that representations for this system based on H can be induced in an appropriate way to yield representations of the original system based on G.

In view of the proliferation of different types of systems to which the method of induced representations is applicable, it is desirable to find a single framework within which most of these types of systems can be handled simultaneously. Such a framework was developed by Fell in an earlier memoir [7], and involves the notion of a Banach $*$-algebraic bundle over a group. By this Fell means a bundle, B, over a locally compact group G, whose fibers, B_x for $x \in G$, are Banach spaces, and on which one has a globally defined multiplication and involution such that

$$B_xB_y \subseteq B_{xy} \quad \text{and} \quad (B_x)^* \subseteq B_{x^{-1}}$$

for $x, y \in G$, satisfying natural algebraic and continuity properties. All of the various types of systems which have been considered seem to be able to be viewed as Banach *-algebraic bundles, except in some nonseparable situations where questions concerning the existence of nice cross-sections are still unresolved. Fell showed in his memoir that for Banach *-algebraic bundles satisfying a condition he called homogeneity (satisfied by all the important examples) the basic theorems in the theory of induced representations could be established.

Fell states in the notes under review, that their main purpose is to show that the condition of homogeneity is not needed for a good part of the theory. But actually he accomplishes much more, in that the general approach and the proofs which he employs are quite different from those of his memoir, in ways which make the theory conceptually clearer and simpler. This is done by taking a quite algebraic approach, along the lines used by the reviewer [23] in reformulating Mackey's definition of induced representations for locally compact groups as a change of ring operation, generalizing Higman's observation. Specifically, if A and B are Banach *-algebras, if X is an A-B-bimodule, and if V is a Hermitian B-module (that is, the Hilbert space of a nondegenerate *-representation of B), then the A-module $X \otimes_B V$ has no canonical inner-product to make it a Hilbert space unless one has some additional structure. What is needed is a B-valued inner-product, $\langle\ ,\ \rangle_B$, on X, since then one can define the inner-product of elementary tensors in $X \otimes_B V$ by

$$\langle x \otimes v, x_1 \otimes v_1 \rangle = \langle \langle x_1, x \rangle_B v, v_1 \rangle.$$

Extending by linearity, one obtains a pre-inner-product on $X \otimes_B V$, whose completion, under suitable hypotheses, will be a Hermitian A-module. This is the construction, in this context, of induced representations, and it is this approach which Fell applies to Banach *-algebraic bundles in these notes. In fact, the first of the three chapters of Fell's notes is devoted to developing a somewhat more general approach to this inducing process in terms of what Fell calls operator inner-products, that is, inner-products on a linear space X, whose values are operators on a Hilbert space (with no specific algebras involved).

The second chapter, which is quite independent of the first, is devoted to developing the general theory of Banach *-algebraic bundles, and it contains a variety of new technical results not contained in Fell's earlier memoir. The third and final chapter then combines the first two chapters to develop the theory of induced representations for Banach *-algebraic bundles.

For locally compact groups the most important of the basic theorems about induced representations is the imprimitivity theorem, which gives an answer to the question "Given a group G and a subgroup H, which representations of G arise by inducing representations of H up to G?". To see what the answer might be, consider first the situation for rings. If S is a subring of the ring R, how can one characterize the R-modules which are of the form $R \otimes_S M$ for some S-module M? To find a special property of such R-modules, let $E = \mathrm{End}_S(R)$, the ring of endomorphisms of R viewed just as a right S-module. Of course, R can be identified as a subring of E by associating to every element of R the endomorphism consisting of left multiplication by that element. Then it is clear that $R \otimes_S M$ will in fact be a left E-module, with the action of E being an extension of the original action of its subring R on $R \otimes_S M$. Thus a necessary condition that a left R-module N be induced from S is that the action of R on N can be extended to an action of E on N. Whether this condition is also sufficient is related to the question of whether the passage from S-modules M to the corresponding E-modules $R \otimes_S M$ is an equivalence from the category of left S-modules to the category of left E-modules. Now the question of when two rings have equivalent categories of modules was first systematically studied by Morita **[21]** (see also **[2]**), and he showed that it was closely related to the existence of bimodules (such as the E-S-bimodule R above) satisfying suitable properties. In the case in which R and S are the group algebras, $L(G)$ and $L(H)$, of a finite group G and a subgroup H, these properties are satisfied. Thus it is the case that an $L(G)$-module is induced from $L(H)$ if and only if the action of $L(G)$ can be extended to an action of E. Furthermore, in this group case, E has a nice description **[24]**. Specifically, let $L(G/H)$ denote the algebra of functions from G/H into the field L with pointwise multiplication. Now G acts by left translation on G/H, and this gives an action of G as algebra automorphisms of $L(G/H)$. Then E is just the semidirect product algebra for this action, so that E-modules correspond exactly to the "covariant representations" of the pair $(G, L(G/H))$. One concludes thus **[24]** that an $L(G)$-module, N, is induced from H iff it can also be made into an $L(G/H)$-module in such a way that it gives a "covariant representation" of $(G, L(G/H))$.

The corresponding imprimitivity theorem for locally compact groups was first formulated and proved by Mackey **[18]**, **[20]** (see also **[23]**), and can be phrased as follows: Let G be a locally compact group and H a closed subgroup. Let $B = C_\infty(G/H)$ denote the C^*-algebra of continuous complex-valued functions on G/H vanishing at infinity, with pointwise multiplication. Then the natural action of G on G/H gives an action of G as a group of automorphisms of B, so that (G, B) is a covariant system. Mackey's imprimitivity theorem states that a representation U of G is induced from H if and only if there exists a representation π of B on the Hilbert space for U such that (U, π) is a covariant representation of (G, B). The culmination of Chapter three of Fell's notes is the generalization of this theorem to the setting of Banach *-algebraic bundles, based on a quite abstract imprimitivity theorem which is the main focus of the first chapter. The statement for bundles is quite similar to that indicated above for locally compact groups.

Fell's approach in terms of bundles has some great advantages over those used in various other papers concerned with establishing a general framework **[3]**, **[16]**, in that Fell can work everywhere with continuous functions, thus avoiding many messy measure-theoretic arguments, and he has no need to become entangled in lengthy cocycle computations and the like. On the other hand, in many specific situations which one may want to study, the bundle structure is often not entirely evident, so that translation between the immediately evident structure and Fell's bundle structure may be tedious. Thus while the theory developed by Fell in these notes is of very considerable philosophical comfort, more experionce will be needed before it will be clear exactly how incisive a technical tool it is for dealing with specific examples.

Fell's notes end with an appendix containing an ingenious proof by Douady and Dal Soglio-Hérault establishing that any Banach bundle over a paracompact or locally compact base space has enough continuous cross-sections. This is surely a basic result.

It is clear that the audience for whom Fell has written these notes consists of specialists in the area, who already understand well the importance of induced representations. Fell includes no examples of induced representations of specific groups ([17] gives a nice survey of examples) and the various other examples which Fell gives either serve as counterexamples, or are novel examples of Banach *-algebraic bundles which would not be familiar to most specialists. On the other hand, the technical, as distinguished from motivational, prerequisites are fairly minimal, involving essentially the rudiments of the representation theory of locally compact groups and Banach *-algebras. Thus these notes are quite self-contained (with the unimportant exception that in various places the reader is referred to Fell's earlier memoir for those proofs which carry over essentially without change). Fell's exposition is very clear. There are myriad details which must be verified, but the overall architecture is quite apparent, and the lines are quite clean.

In what directions is this subject likely to develop during the next decades? The most glaring gap is probably the lack of any general theory of representations of groups which are not locally compact. Representations of such groups are becoming increasingly important in physics, for example in current algebra [10] and gauge theories, and there have recently appeared a scattering of articles containing fascinating information about the unitary representations of certain specific groups which are not locally compact ([1], [9], [14], [15] and references therein). Induced representations will undoubtedly be of importance, and presumably applications to physics will involve covariant representations and very possibly corresponding generalizations to bundles.

There are two preprints, by Dang Ngoc [4] and Green [11], which have appeared since Fell's notes went to press, and which are worth mentioning here, since they introduce yet another type of system which can be treated within Fell's framework, but which is itself technically quite useful. This consists of a covariant system (G, A) together with a normal subgroup, N, of G and a homomorphism of N into the unitary group of the double centralizer algebra of A which is appropriately compatible with the action of G on A. Dang Ngoc calls this a "produit croisé restreint", and Green a "twisted covariance algebra". Green, for example, is able to use this structure to simplify, generalize, and illuminate the penetrating work of Pukanszky on characters of connected Lie groups [22].

In summary, Fell has done a fine job of providing specialists in the subject with an elegant general framework for studying unitary induced representations, in a context wide enough to cover just about any system involving a locally compact group and a Banach *-algebra which is likely to be of serious interest.

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Topological algebras, by E. Beckenstein, L. Narici, and C. Suffel, North-Holland, Amsterdam, New York, Oxford, 1977, xii + 370 pp., $24.50.

An algebra, in the present context, is an associative linear algebra over a field K, and a topological algebra is such an algebra with a suitably related topological or quasi-topological structure. This statement requires, of course,