BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 1, January 1979

0002-9904/79/0000-0014/$01.75

BOOK REVIEWS

Abstract analytic function theory and Hardy algebras, by Klaus Barbey and Heinz König, Lectures Notes in Math., vol. 593, Springer-Verlag, Berlin, Heidelberg, New York, 1977, vii + 256 pp.

The main purpose of this set of notes is to provide a format for a unified exposition of the research of Heinz König. The first chapter recounts some standard theorems about harmonic and holomorphic functions in the unit disc. These provide the background for what follows. The goal is to take each of these results and show that it is, to a greater or lesser degree, a special case of a general principle about algebras of functions. Of course this approach did not originate with the authors of this set of notes (nor do they claim that it does). Perhaps what distinguishes their work from the work of others is their success in avoiding, where possible, special assumptions on the algebras of functions (i.e. that they be Dirichlet algebras, etc.).

The F. and M. Riesz theorem in one of its classical forms, says that if $d\mu$ is a measure on the circle, T, and $\int z^n\, d\mu = 0$, for $n = 0, 1, 2, \ldots,$ then $d\mu$ is absolutely continuous with respect to Lebesgue measure dm. It was realized at some point that this conclusion would follow rather easily if it could be shown that if $d\mu = h\, dm + d\nu$ is the Lebesgue decomposition of $d\mu$ with respect to dm, then $\int z^n h\, dm = 0$, $n = 0, 1, \ldots.$ In this form the F. and M. Riesz theorem becomes a statement about the Lebesgue decomposition of a measure on T that annihilates the algebra, A, of holomorphic polynomials. The authors' version proceeds from this point of view; (X, Σ) is a measurable space and A is a subalgebra of the algebra, $B(X, \Sigma)$, of bounded measurable functions on (X, Σ). It is assumed that $1 \in A$. By $\Sigma(A)$ is meant the set of nonzero complex homomorphisms, φ, of A for which there is a complex measure $\mu \in M(X, \Sigma)$ such that $\varphi(f) = \int f\, d\mu$ for all $f \in A$; $M(\varphi)$ is the set of all probability measures $m \in M(X, \Sigma)$ such that $\varphi(f) = \int f\, dm$, for all $f \in A$. (It is shown that $M(\varphi) \neq \varnothing$.) Using the theory of "bands" of measures it is shown that if $\mu \in M(X, \Sigma)$ then $\mu = \mu_1 + \mu_2$, where μ_1 is absolutely continuous with respect to some element of $M(\varphi)$ and μ_2 is singular with respect to every element of $M(\varphi)$. The abstract F. and M. Riesz theorem says that if μ annihilates A then the same is true of μ_1 and μ_2.

The decomposition of annihilating measures is extended further by introducing the notion of "Gleason part" into this context. By $M(\varphi)^{\vee}$ is meant the set of measures on (X, Σ) which are absolutely continuous with respect to some element of $M(\varphi)$. It is then shown that if $\varphi, \psi \in \Sigma(A)$, then either $M(\varphi)^{\vee} = M(\psi)^{\vee}$ or $M(\varphi)^{\vee} \cap M(\psi)^{\vee} = \{0\}$. This establishes an equivalence relation on $\Sigma(A)$; the equivalence classes are called the Gleason parts of $\Sigma(A)$; the set of equivalence classes is denoted by $\Gamma(A)$. It is then shown that if μ annihilates A then $\mu = \sum_{P \in \Gamma(A)} \mu_P + \mu_S$, where each μ_P annihilates A and each μ_P is absolutely continuous with respect to some element of $M(\varphi)$ where $\varphi \in P$, and μ_S is singular with respect to every element of $M(\varphi)$ for every $\varphi \in \Sigma(A)$.

This abstract F. and M. Riesz theorem is the first result on "abstract analytic function theory" presented in these notes. It sets the stage for much of what follows. The point is made that we are naturally led to focus our attention on a single $m \in M(\varphi)$ for some $\varphi \in \Sigma(A)$. This gives rise to the somewhat more general notion of a Hardy algebra. A Hardy algebra is a weak * closed subalgebra of $L^\infty(dm)$ for some finite positive measure space (X, Σ, dm). It is assumed that $1 \in H$ and that there is a nonzero complex homomorphism, φ, of H that is weak * continuous. There follow three chapters in the notes devoted to the theory of Hardy algebras. I will discuss some of the results briefly.

Instead of the usual family of H^p spaces, a single function class, $H^\#$, is introduced. A function f is said to belong to $H^\#$ if there is a sequence $\{u_n\}$, $u_n \in H$, $\|u_n\|_\infty \leqslant 1$, $u_n \to 1$ a.e., dm, and $u_n f \in H$, for all n. Now $H \subseteq H^\#$ and $H^\#$ is an algebra. If this construction is carried out for $H = H^\infty$, the algebra of bounded holomorphic functions in the unit disc, then $H^\#$ will be the set of functions of the form u/v, where $u, v \in H^\infty$ and v is outer. This is sometimes called the Smirnov class, or N^+. In the general situation a theorem is proved giving necessary and sufficient conditions in order that a positive measurable function be the modulus of an invertible element of $H^\#$. A special case is the classical theorem that a positive L^1 function on the circle is the modulus of an H^1 function if and only if it has an integrable logarithm. There is also a chapter on conjugate functions. An abstract conjugation is defined and estimates of Kolmogoroff and M. Riesz type are proved.

After the theory of Hardy algebras is developed there follow four chapters on special topics. I found the chapter on the Mooney-Havin theorem especially interesting. In the classical case, the theorem in question states that if $\varphi_n \in L^1(T)$ and if

$$\lambda(f) = \lim_{n\to\infty} \int_T f\varphi_n \, dm$$

exists for every $f \in H^\infty$ then there is a $\varphi \in L^1(T)$ such that $\lambda(f) = \int f\varphi \, dm$. Specialized to the classical case the proof of Barbey and König goes like this. Using standard results one sees that there is a $\varphi \in L^1(dm)$ such that $\lambda(f) = \int f\varphi \, dm + \tau(f)$, where τ is singular, i.e., there are sets $E_n \subseteq T$, such that $E_{n+1} \subseteq E_n$, $m(E_n) \to 0$, and $\tau(f) = \tau(fX_{E_n})$ for all $f \in L^\infty$ and all n. By replacing φ_n by $\varphi_n - \varphi$ we may assume that λ itself is singular and the aim is to show that $\lambda(f) = 0$ for all $f \in H^\infty$. One finds $h \in H^\infty$ such that $\operatorname{Re} h \geqslant c_n$ on E_n and $c_n \to \infty$ as $n \to \infty$. Now for every $t > 0$,

$$|fX_{E_n}/(1 + th)| \leqslant \|f\|_\infty/(1 + tc_n)$$

and so

$$|\lambda(f/(1 + th))| \leqslant \|\lambda\| \, \|f\|/(1 + tc_n).$$

Letting $n \to \infty$ it follows that $\lambda(f/(1 + th)) = 0$ for every $t > 0$. One would like to let $t \to 0$ and conclude that $\lambda(f) = 0$. Now $f/(1 + th) \to f$ pointwise and boundedly, which isn't good enough since λ is not absolutely continuous. Also it isn't necessarily the case that $f/(1 + th) \to f$ uniformly. However it is true that $f/(1 + th) \to f$ in a sense which is weaker than uniform convergence but stronger than bounded pointwise convergence. A sequence $f_n \in L^\infty$ is said to converge strictly to $f \in L^\infty$ if $f_n \to f$ pointwise and $\Sigma |f_{n+1} - f_n| \in L^\infty$. Strict convergence is stronger than bounded pointwise convergence so any weak * closed subspace of L^∞ is closed under strict convergence. If $S \subseteq L^\infty$ is a weak * closed subspace and Λ is a linear functional on S, Λ is called strictly continuous if whenever $f_n \to f$ strictly then $\Lambda(f_n) \to \Lambda(f)$. It is clear that any linear functional Λ, where $\Lambda(f) = \int f\varphi \, dm$ with $\varphi \in L^1$ is strictly continuous. The proof of the Mooney-Havin theorem now follows from two key facts. (i) If $\{\Lambda_n\}$ is a sequence of strictly continuous linear functionals on a weak * closed subspace $S \subseteq L^\infty$ and if $\Lambda(f) = \lim_{n\to\infty} \Lambda(f_n)$ exists for all $f \in S$ then Λ is strictly continuous; (ii) if $t_n \to 0$, $t_n > 0$, then $f/(1 + t_n h) \to f$ strictly.

There are many other topics covered in these notes that I have not mentioned. For example there is a chapter on imbedding analytic discs and a chapter on rational approximation.

The material is well organized and carefully presented. Many of the proofs are extremely elegant.

PATRICK AHERN

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 1, January 1979

0002-9904/79/0000-0013/$02.75

The theory of partitions, by George E. Andrews, in Encyclopedia of Mathematics and its Applications, volume 2, Addison-Wesley Publishing Company, Advanced Book Program, London, Amsterdam, Don Mills, Ontario, Sydney, and Tokyo, 1976, xiv + 255 pp., $19.50.

The serious study of partitions probably started when Euler was asked how many ways fifty could be written as the sum of seven summands. From this modest beginning a beautiful field has grown up that has connections with a number of different areas of mathematics.

Ferrers, in a letter to Sylvester, observed that it was possible to represent a partition by an array of dots. For example, $7 = 4 + 2 + 1$ is represented by

$$\begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \\ \cdot & & & \end{array}$$

A large number of identities can be proved by suitably counting the dots in a Ferrers graph. One beautiful example is F. Franklin's proof of the following result of Euler.

Let $P_n(D, e)$ denote the number of partitions of n into an even number of distinct parts and $P_n(D, o)$ the number of partitions of n into an odd number of distinct parts. Then

$$P_n(D, e) - P_n(D, o) = \begin{cases} 0, & n \neq k(3k \pm 1)/2, \\ (-1)^k, & n = k(3k \pm 1)/2, k = 0, 1, \ldots. \end{cases} \tag{1}$$

This proof is given in Chapter 1 and anyone who is interested in seeing how mathematics can be done without having to introduce many definitions