evident in research on group representations. Notions from homological algebra and algebraic K-theory have clarified many features of the modular theory, as well as the difficult integral representation theory, which deals with representations over various types of integral domains. Work goes on on all parts of the subject, and there is still a great deal to be discovered.

Serre's book gives a fine introduction to representations for various audiences. It is divided in three parts. The first was originally an appendix to a book on quantum chemistry by Gaston Berthier and Josiane Serre. It gives an exposition of the basics of complex characters and representations, in a style suitable for nonspecialists. There are also a few remarks on the extension of the theory to compact groups.

The second part is for a more sophisticated reader. It gives more detailed information on complex characters, and then proceeds to deeper topics. These come under two main headings. First, there is a discussion of induction theorems, which tell when characters of a group can be obtained in a natural way from characters of certain subgroups. Second, rationally questions in characteristic zero are considered. Thus, one sees what happens when the complex field is replaced by a subfield which may be too small to realize all the complex representations.

The third part is an exposition of Brauer's modular theory. Here, categorical notions (projective covers, Grothendieck groups) are used freely. The connection between complex, integral and modular representations is examined very elegantly, and the Fong-Swan Theorem on lifting modular characters of p-solvable groups is obtained as an application. The Brauer characters are discussed briefly, but block theory is omitted altogether.

Despite the brevity of the book and its omission of many topics, the specialist can profit greatly from reading it. As always with Serre, the exposition is clear and elegant, and the exercises contain a great deal of valuable information that is otherwise hard to find. Also, the discussion of rationality questions is by far the best available. The translation, by L. L. Scott, Jr., is excellent; the design and typography are up to Springer-Verlag's superb standards. Thus, although the book is no substitute for the encyclopedic works of Curtis and Reiner and of Dornhoff, it is highly recommended for specialists and nonspecialists alike.

W. H. GUSTAFSON

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 5, September 1978

Applied and computational complex analysis. II, by Peter Henrici, Wiley, New York, London, Sydney, Toronto, 1977, ix + 662 pp., $32.50.

What would you put into a text for a second course in complex analysis? I expect that most of us, faced with this decision, would follow Hille in accepting some material as canonical and pursue our personal interests for the rest. Hille's basic list consisted of analytic continuation, elliptic functions, entire and meromorphic functions, normal families, and conformal mapping, but was for a rather "pure" course. Suppose it is to be a course oriented toward applications, meaning applications outside of mathematics itself? One has to consider what the applicable parts of the subject are (now, not in some imagined millennial future). Certainly residue theory and conformal mapping are applicable, and more of these is needed than was in the first course; asymptotic evaluation of integrals by steepest descents; complex inversion of Laplace transforms; probably Wiener-Hopf techniques or more generally the Riemann-Hilbert boundary value problem. From Hille's canonical list, besides conformal mapping, elliptic functions probably count as applicable, and so do entire functions now that they have come into prominence in communication theory and optics.

Henrici continues in the style of his first volume by offering rather different answers. His chapter headings are Infinite products, Ordinary differential equations, Integral transforms, Asymptotic methods, and Continued fractions. Of course entire functions are hiding in the chapter on infinite products, and again among the Laplace transforms; Laplace transforms are just one kind of integral transform, and asymptotic methods are as expected; but where are the other desiderata, and what are differential equations and continued fractions doing in this volume? Conformal mapping has been officially deferred to Volume 3 as a technique for solving partial differential equations; some of the other omitted topics are concealed under the very broad chapter headings.

The separate chapters are effectively short books in themselves, and will be discussed as if they actually were separate. The chapter numbering continues from Volume 1, reviewed in Bull. Amer. Math. Soc. **81** (1975), 647–652.

8. Infinite products. I suppose that every beginning mathematician, after being introduced to infinite series, asks, "what about infinite products?" Of course there *are* infinite products; they play a smaller role, perhaps because their theory can be reduced to the theory of series. There are, however, advantages to having a separate theory of products, because they are useful in several contexts. For example, a power series is a natural generalization of a polynomial, but power series are notoriously uninformative about the zeros of their sums. The natural generalization of a polynomial written as a product of linear factors is an infinite product that displays the zeros (although the proper generalization is less straightforward than this remark might suggest).

This chapter is short (only 74 pages), but it contains everything that a beginning student needs to know, and a great deal else that is not ordinarily thought of as textbook material. Henrici begins by showing us some of the famous products connected with partitions of integers. We return to more standard material with the gamma function, including a deduction of Stirling's formula from the infinite product. Gamma functions lead to hypergeometric functions, and these are then pursued to the end of the chapter for their own sake; far enough to get some of the famous identities, as well as Pochhammer's representation and Mellin-Barnes integrals, which illustrate some of the more sophisticated applications of contour integration.

9. Ordinary differential equations. Given the title of the book as a whole, it is not surprising that this chapter is almost entirely about linear differential equations in the complex domain. The theory of these equations is one of the classics of analysis. The algorithmic method does not have much new to say here since the whole subject has always been a bunch of algorithms, not always perfectly understood. Most mathematics majors, and certainly almost all students of applied mathematics, are familiar with the techniques of solving the equations by using formal series, but most textbook presentations are apt to leave the student somewhat mystified. The techniques seem to work in the real domain where differential equations usually live; why, the student wonders, are we to be interested in what is going on in the complex plane? One answer, of course, is that we cannot understand the rationale of the techniques if we stay in the real domain, any more than we can understand the convergence theory of power series by staying in the real domain.

In this chapter everything is done with care, and systems of equations are treated by matrix methods, as is proper. Bessel functions and the confluent hypergeometric function make brief appearances; the role of Fuchs' conditions is made clear; then we meet Riemann's theory of the hypergeometric equation, complete with quadratic transformations; the whole theory of Legendre's equation drops out as a very special case. (One doubts, however, that a working physicist would appreciate being asked to go this route in order to be able to solve three-dimensional problems with spherical symmetry.)

This is a clean treatment in modern language, and it seems clear that the material really does belong (where it is seldom seen nowadays) in a course on complex analysis.

10. Integral transforms. The word "transforms" is, if one takes a very abstract position, simply a synonym for "function": to transform something is to use it as the argument of a function. Of course "integral transform" has special connotations, particularly in the present context: to form an integral transform one multiplies a given function by a "kernel" and integrates, thus producing a new function that has (one hopes) desirable properties. Henrici starts out with Laplace transforms (kernel e^{-zt}, $0 < t < \infty$), from the very pragmatic point of view of wanting to solve nonhomogeneous linear differential equations with constant coefficients. To a considerable extent, the intrinsic charm of the material then takes over, and in the latter part of the chapter the differential equations recede into the background.

The idea of using Laplace transforms to solve differential equations is like a more complicated version of using logarithms to do multiplication: we transform the problem into a more tractable one, and then transform the solution back. (In fact, one could present logarithmic multiplication explicitly as an example of the use of Laplace transforms.) The key observation, for differential equations, is that $(\partial/\partial x)e^{-xt} = -te^{-xt}$, which shows how a differential operator is going to be transformed into a polynomial. The same sort of thing can be done by symbolic (or operational) methods, but these involve dangers that for many years only really good scientists seemed able to avoid consistently. The Laplace transform provided a satisfactory and reliable translation of operational calculus; one might say that Heaviside was to operational methods what Napier was to logarithms (remember that the interpretation of logarithms as exponents came later). However, whereas logarithmic multiplication is dead (killed by the hand-held electronic calculator), the Laplace transform is very much alive, so much so that some engineers seem to think with transforms in preference to the original functions. Of course the Laplace transform has other uses, but it is as a device for solving differential equations that it makes its appearance in this chapter. The theory is discussed fully but succinctly, and the discussion of solving differential equations extends to systems (often short-changed in textbooks as being too difficult); there is also a convenient glossary of the vocabulary of systems engineering: transfer functions and all that. Henrici introduces Fourier transforms primarily to get at the complex inversion for Laplace transforms, but he includes other applications of Fourier series and integrals, including the Poisson summation formula. Laplace integrals lead into Dirichlet series and thence to a full proof of the prime number theorem (which appears to have little to do with either algorithms or applications, unless one accepts Wiener's dictum that the problem of the distribution of the primes is equivalent to a problem of analyzing black body radiation).

Something quite unusual for a textbook follows: the Laplace transform for entire functions of exponential type and Pólya's theory of the indicator diagram, with extensions to functions of exponential type in an angle. There is a substantial section on the discrete Laplace transform, $LF(s) = \eta\Sigma_{n=0}^{\infty} e^{-sn\eta}F(n\eta)$. (The special case $\eta = 1$ is known to engineers as the z-transform.) If we write $t = e^{-\eta s}$ a discrete Laplace transform is nothing but the generating function for the sequence $\{\eta F(n\eta)\}$ (in the sense in which the term is used in number theory and probability; many writers use "generating function" in the extended sense of "Laplace transform"). Treating the discrete Laplace transform in the framework of Laplace transforms rather than that of power series presumably has advantages if you are already familiar with the Laplace transform in general; no doubt the principle could be pushed even further. Henrici uses it, among other things, to prove Carlson's theorem on functions that vanish on an arithmetic progression. The chapter ends with some notes on other integral transforms and some examples of partial differential equations solved by transforms.

11. Asymptotic methods. An asymptotic method can mean any method for obtaining approximate representations of functions, but usually it refers to a representation that involves a parameter and becomes more useful as the parameter gets closer to a limit; Stirling's formula for the gamma function is an example.

The most usual asymptotic representations use power series (generally in powers of $1/z$, since traditionally one deals with a "large" parameter), but these are not the convergent power series of elementary work. These power series generally diverge at every point–but convergence isn't everything. Convergence says that the approximation by partial sums of order n can be made very good at a given z by taking n large enough (without specifying how large). On the other hand, for an asymptotic series the approximation generally deteriorates for a given z if n is large, but if n is held fixed and z gets large (the larger, the better), the approximation improves–not in the sense of a small difference between the function and its approximations, but in the sense that the ratio between the function and its approximations approaches 1 as $z \to \infty$. It is in this sense that one has to interpret the remark, attributed to a well-known physicist (I forget which one), "Fortunately this series diverges, so we can use it." What he meant was that whereas convergent series often converge so slowly as to be useless for numerical work, appropriate partial sums of a divergent (asymptotic) series can on occasion give very accurate approximations to a function associated with the series. A precise definition of what one should mean by an asymptotic power series was given by Poincaré; there are similar definitions for other series with similar properties.

This chapter presents several topics. To begin with, asymptotic power series are applied to the solution of ordinary differential equations with singular points. Here we meet the Stokes phenomenon, sometimes described as "the discontinuity of arbitrary constants," which is that the form of the asymptotic representation of a solution will change discontinuously as we go from one sector of the complex plane to another, although the solution itself is perfectly well behaved. (One of the properties of convergent power series that asymptotic power series lack is the property of providing equally good approximations in all directions.) The Stokes phenonenon was hard for the formalists of the nineteenth century to swallow, and it still has some shock value. As one would expect, Henrici gives a lucid discussion of the method of steepest descent for integrals containing a large parameter; examples: gamma and beta functions, Bessel functions, hypergeometric functions. Less familiar examples of asymptotic representations include series with x^n replaced by $(z-k)_n/n!$, where $(z)_n$ means $z(z+1)\cdots(z+n-1)$. The chapter ends with the Euler-Maclaurin formula and Romberg's algorithm for numerical evaluation of limits; the first is very old; the second, very modern. Some of this material is complex analysis only by courtesy, but it all fits together to provide a general survey (less detailed than one might wish for) of asymptotic representations.

R. P. BOAS

12. Continued fractions. With its 170 pages this is by far the longest chapter in the second volume. Nevertheless there is just barely enough material in it to whet one's appetite concerning the analytic theory of continued fractions. There is nothing encyclopedic about the treatment. The emphasis is on a leisurely style of exposition. Most of the results presented are classical but the way they are discussed usually differs from the standard approach and frequently opens up new vistas and relations to other topics. Sequences of Moebius transformations

$$T_n(z) = t_1 \circ t_2 \circ \cdots \circ t_n(z),$$

where

$$t_n(z) = a_n/(b_n + z),$$

are used to define continued fractions as triples $\langle\{a_n\}, \{b_n\}, \{T_n(0)\}\rangle$. Another approach is to define the *approximants* $T_n(0) = A_n/B_n$ by means of the second order linear difference equations

$$A_n = b_nA_{n-1} + a_nA_{n-2}, \qquad B_n = b_nB_{n-1} + a_nB_{n-2},$$
$$A_0 = 0, \quad A_1 = a_1, \quad B_0 = 1, \quad B_1 = b_1.$$

The former more geometric approach, though not unknown earlier, was pushed into the background by Pringsheim's and Perron's preference for the development based on the difference equations. In recent years however the geometric approach has been chosen by most workers in the field. Henrici shares this preference.

There are two major areas to which continued fractions have been applied. The number theoretic applications depend for the most part on the regular (or simple) expansion of irrational numbers x, $0 < x < 1$ into infinite continued fractions. One defines

$$x_0(x) = x, \quad x_n(x) = 1/\left(x - \left[x_{n-1}(x)\right]\right), \quad b_n(x) = \left[x_n(x)\right],$$

where $[a]$ denotes the integral part of a. The *regular continued fraction expansion* of x is then

$$K(1/b_n(x)) = \frac{1}{b_1(x)} + \frac{1}{b_2(x)} + \frac{1}{b_3(x)} + \cdots.$$

Its sequence of approximants $\{p_n(x)/q_n(x)\}$ always converges to x. Moreover $p_n(x)/q_n(x)$ provides the best rational approximation to x in the sense that

$$|bx - a| > |q_n(x)x - p_n(x)|,$$

provided a and b are integers relatively prime to each other and $0 < b \leqslant q_n(x)$. By means of continued fractions one can also prove that

$$|x - p/q| < 1/q^2\sqrt{5} \qquad (*)$$

has infinitely many rational solutions p/q, $q \neq 0$, $(p, q) = 1$. This is so since among any three consecutive approximants of the regular continued fraction expansion of x there is at least one satisfying the inequality $(*)$. There are also results with a measure theoretic flavor. We give two samples.

(a) For almost all x, $0 < x < 1$, the sequence of elements $\{b_n(x)\}$ is unbounded.

(b) For almost all x, $0 < x < 1$, the inequality

$$|x - p/q| < 1/q^2 \log q$$

has an infinite number of solutions p/q, $q \neq 0$, $(p, q) = 1$.

The second and even richer area of applications is in complex variable theory. It is with this that the book under review is mainly concerned. Here the central problem is the expansion of holomorphic functions into various kinds of continued fractions. Since such functions always have Taylor series expansions the question arises: what can be gained by looking for continued fraction expansions. The answer is that:

(α) The continued fraction may converge in a larger region than the Taylor series. (In particular the function in question may only be given by an asymptotic series, say at ∞, while the corresponding continued fraction will converge in a region having ∞ on its boundary.)

(β) It may be easier to obtain information about the value behavior of the function from the continued fraction than from the Taylor series.

(γ) The continued fraction may converge faster and an adequate approximation may be easier to compute.

Topics in which some or all of these considerations have played a role are:

Expansions of hypergeometric functions, Padé tables and the Stieltjes moment problem which is solved by obtaining a continued fraction expansion of a function defined by an integral transform. (Note how several of the topics from earlier chapters of the book are intertwined with the analytic theory of continued fractions.)

An essential part of the analytic theory is an ongoing investigation of the convergence behavior of continued fractions. Particular emphasis has recently been placed on an analysis of truncation errors and speed of convergence as well as computational stability of various algorithms by means of which continued fractions can be evaluated. This is in part in response to the fact that modern computers can handle continued fractions with ease so that the subject has become of practical importance.

Henrici gives a modern proof of the classical result of Worpitzky (1865) that $|a_n| \leqslant 1/4$, $n \geqslant 1$, is sufficient for the convergence of $K(a_n/1)$. Indicative of the progress that has been made since then is the result, obtained in 1942, that for any α, $-\pi/2 < \alpha < \pi/2$ and any $M > 0$,

$$|a_n| - \operatorname{Re}(a_n e^{-2i\alpha}) \leqslant 1/2\cos^2\alpha, \qquad n \geqslant 1,$$

together with $|a_n| < M$, $n \geqslant 1$, is sufficient for the convergence of $K(a_n/1)$. Convergence theory for continued fractions differs markedly from the situation for infinite series and products. In particular most convergence criteria are of the *convergence region* type (as in the case of the circular disk and parabolic regions mentioned above) and in the proofs important information is obtained about the values taken on by the approximants.

My co-reviewer suggests that inclusion of a chapter on continued fractions is unusual in a modern book on complex analysis. This is indeed the case. However, it was not always so. The books by Stern (1860), Stolz (1885), and Pringsheim (1921) all contained long sections on continued fractions.

As far as notation is concerned the author goes very much his own way. Thus on the one hand he uses the European notation

$$\frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots,$$

instead of

$$\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots,$$

which is customary in this country. On the other hand he uses $\Phi(a_n/b_n)$ (in analogy to Σ and Π for series and products, respectively) instead of $K(a_n/b_n)$ (first suggested by Knopp, the K standing for Kettenbruch) which is quite widely used.

As in many other mathematical disciplines colorless terms such as "regular", "normal", "associated", "corresponding", etc. are frequently used in the theory of continued fractions. In addition we have C-fractions, J-fractions, P-fractions, and T-fractions among others. Unless Bourbaki can be persuaded to write a treatise on our subject it is unlikely that there will be a change in our drab terminology soon. Henrici coins such new terms as RITZ and SITZ. Though an explanation is provided for the choice of these terms it