considered: Is every Hausdorff compactification of a Tychonoff space (a) a Wallman-type compactification? (b) a GA compactification (defined by DeGroot and Arts)? Since every Wallman-type compactification is a GA compactification, the questions are related. In a footnote the author mentions that (a) has supposedly been answered in the negative by Uljanov and Shapiro. Conditions are given for a superextension to be a regular supercompact space, and hence a superextension of each dense subspace.

The final chapter presents a summary of recent results on supercompact spaces which answer some of the questions posed in earlier chapters.

The monograph concludes with an extensive bibliography and an author reference index as well as a subject index. For the topologist interested in extension theory, this book provides a good insight into current research in the area of supercompactness. The author has done an excellent job of bringing together diverse results which all contribute to the general theory of supercompactness, and should be extremely valuable to anyone contemplating research in this area.

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Selected mathematical papers of Axel Thue, edited by Trygve Nagell, Atle Selberg, Sigmund Selberg, and Knut Thalberg, Universitetsforlaget, Oslo, LVIV + 591 pp., $40.00.

The present volume, edited by T. Nagell, A. Selberg, S. Selberg and K. Thalberg, contains Thue's papers in the theory of numbers and in mathematical logic. Thue's papers on geometry and mechanics are not included, but there is a possibility that they will be collected in a later volume.

The publication of these Selected Papers is a great service to the mathematical community. Thue's name is known mostly for his theorem on diophantine approximation and on diophantine equations which appeared in 1908/1909. The present selection of his work destroys the common myth (long held by the reviewer) that this was Thue's only important contribution.

The volume contains a biography of Thue by Brun who knew Thue, there is an introduction to Thue's work in number theory by Siegel, and there is an article by Siegel which gives a deep analysis of Thue's work on approximation to algebraic numbers. These contributions almost preempt the task of the reviewer.

Thue liked to work independently, and his papers require almost no prerequisites. It appears that Thue was not much influenced by Lie, Kronecker and other great mathematicians with whom he had contact during his years of study in Leipzig and Berlin. His greatest work, on approximation to algebraic numbers, appeared in 1908/1909, when he was well in his forties, and when he had been away from the centers of mathematics for over a decade. There is an intriguing picture on a front page, with Thue gazing rather sadly into the distance.

As Siegel points out (in his (1970) article which is reprinted here), Thue wrote nine papers about approximation to algebraic numbers, but only one attracted wider attention. This paper (Thue, 1909) contains the famous result that if ρ is algebraic of degree $d \geqslant 3$, and if $\mu > \frac{1}{2}d + 1$, then there are only finitely many rationals p/q with

$$|\rho - p/q| < q^{-\mu}.$$

An easy consequence is that if $f(x, y)$ is a binary form of degree at least 3 without multiple factors and if $c \neq 0$, then the diophantine equation (nowadays called the "Thue Equation")

$$f(x, y) = c$$

has at most finitely many solutions in rational integers x, y. This latter result was already contained in Thue's (1908c) article, written in Norwegian. Landau called it the most important discovery in "elementary number theory" which he had witnessed during his lifetime. He also said ten years after its publication that already ten competent mathematicians had read Thue's paper.

Suppose $P(x)$ is a polynomial with rational integer coefficients of degree r having a zero of degree s at the algebraic number ρ. If $|\rho - (p/q)| < 1$, then the Taylor expansion of $P(x)$ at $x = \rho$ yields $|P(p/q)| \leqslant c_1|(p/q) - \rho|^s$, with a constant c_1 depending only on P. On the other hand, if p/q is not a root of P, we have $|P(p/q)| \geqslant q^{-r}$, and combining this with the inequality above we obtain $|\rho - (p/q)| \geqslant c_2q^{-r/s}$. By taking P to be the defining polynomial of ρ we obtain $r = d$ (the degree of ρ), $s = 1$ and $|\rho - (p/q)| \geqslant c_2q^{-d}$, which is a theorem of Liouville (1844). It is clear that always $r/s \geqslant d$, so that an improvement is not possible by this approach.

Thue had the far reaching idea of replacing the polynomial above by a polynomial $P(x, y) = yQ(x) - P(x)$ in two variables x, y. If this polynomial is of total degree at most r, then the number of possible coefficients of $P(x)$, $Q(x)$ is $2r + 1$. Thue now wants $P(x, \rho)$ to have a zero of high order s at $x = \rho$. A zero of order s amounts to s linear relations in the coefficients of $P(x)$ and $Q(x)$. Each of these relations has coefficients in the number field of degree d generated by ρ, and hence is equivalent to d linear equations with rational coefficients. So we have altogether sd equations with rational coefficients, and if $sd < 2r + 1$, then there is a nonzero solution to our problem. So we may take $r/s \approx \frac{1}{2}d$, which accounts for Thue's improvement over Liouville's estimate. Thue uses the box principle to construct polynomials $P(x)$, $Q(x)$ whose coefficients are small.

Before using the box principle, and again afterwards, Thue (1908a), (1908b), (1910c), (1919) was able to construct $P(x)$, $Q(x)$ explicitly in certain cases, in particular for $\rho = \sqrt[d]{b}$. The polynomials obtained are hypergeometric, but only Siegel (1937) recognized them to be such. Siegel in his (1970) analysis of Thue's papers notes his surprise that Thue did not recognize the polynomials to be hypergeometric. Siegel then shows that Thue's explicit construction method applies precisely when the Thue equation is of the type $(\alpha x + \beta y)^d + (\gamma x + \delta y)^d = c$ (which includes the general cubic case), and he discovers the amazing fact that the number of solutions of such an equation is below a bound which depends on c and the degree d only. It may be conjectured that this continues to hold for an arbitrary Thue equation.

At this point it is appropriate to give a brief account of subsequent developments of Thue's work on approximation. Siegel, trying to understand Thue's paper, rewrote it, and in the process he replaced the polynomial $yP(x) - Q(x)$ by a more general polynomial in two variables, which enabled him to weaken Thue's condition $\mu > \frac{1}{2}d + 1$ to $\mu > 2\sqrt{d}$. (This was in 1917. Siegel ("Zur Einführung" of the present collection) gives an interesting account of how Schur treated his work unfairly, so that it appeared only in (1921a), (1921b). Siegel also says that he had difficulty understanding Thue's letters $c, k, \theta, \omega, m, n, a, s$. In fact, Thue's papers are sometimes difficult to read, not in the least because often the reader does not know until the end where the investigation is leading to.) Finally Roth (1955) used polynomials in many variables and obtained the condition $\mu > 2$, which earned him the 1956 Field Prize. The inequality $\mu > 2$ is best possible, but it is conjectured that, say, $|\rho - (p/q)| > c_3(\rho)(q^2 \log^2 q)^{-1}$. On the other hand, it is unlikely that $|\rho - (p/q)| > c_4(\rho)q^{-2}$ if ρ is of degree $d \geqslant 3$, since this would imply that ρ has bounded partial denominators in its continued fraction. Schmidt (1970) generalized Roth's Theorem to simultaneous approximation. For example, he proved that if $1, \rho_1, \ldots, \rho_n$ are real algebraic numbers which are linearly independent over the rationals, then for $\mu > n + 1$ there are only finitely many rationals n-tuples $p_1/q, \ldots, p_n/q$ with

$$\left|\rho_1 - \frac{p_1}{q}\right| \cdots \left|\rho_n - \frac{p_n}{q}\right| < q^{-\mu};$$

here $\mu > n + 1$ is best possible. For further references see Schmidt (1971b).

Siegel (1929) used his (1921) work to deduce his famous discovery that a diophantine equation in two variables of positive genus has at most finitely many solutions. Schmidt (1971a), (1972) applied his work on simultaneous approximation to classify norm form equations $\mathfrak{N}(\rho_1 x_1 + \cdots + \rho_n x_n) = c$ in variables $x_1, \ldots, x_n$ which have infinitely many solutions.

Thue needs two distinct rational approximations $p_1/q_1, p_2/q_2$ to substitute into his polynomial $P(x, y)$. Consequently, a contradiction is obtained only if one has two very good rational approximations to ρ. Davenport (1968) and Schinzel (1967) gave explicit constants $\mu^*(\alpha) < d$ and $c^*(\alpha)$ such that $|\rho - (p/q)| > q^{-\mu^*}$ if $q \geqslant c^*$, with one possible exception p/q. But already Thue (1919) and Siegel (1937) proved that under suitable conditions, equations $ax^n - by^n = c$ have at most one solution. If one knows a solution, then there is no other. Thue gives the example $|x^7 - 17y^7| < 10^6$, which has no solution with $x \geqslant 14293$. But in general, Thue's method and the subsequent developments of Siegel, Roth and Schmidt are ineffective, i.e. they do not permit to give bounds for the size of the solutions of inequalities or equations. Baker ((1966) and subsequent papers; in particular (1968)), in work which earned him the 1970 Field Prize, developed a method which is different from Thue's and more akin to that of Gelfond, and which does permit him to give explicit bounds for Thue's equation. Feldman (1971) utilized this method to show that without exception $|\rho - (p/q)| > q^{-\mu'}$ if $q \geqslant c'$, where $\mu'(\alpha) < d$ and $c'(\alpha)$ are computable. Earlier Baker (1964) had used the approach via hypergeometric functions to deal with the case $ax^n - by^n = c$. For further references see Baker (1975).

Most of the results quoted can be carried over to approximation in p-adic fields or power series fields. See Mahler (1941), (1961). A new approach was initiated by Osgood (1973), (1975) in the power series case. Instead of polynomials in two or more variables, Osgood uses differential polynomials $P(x, x', \ldots, x^{(m)})$ (where the derivative is with respect to t if expansions in t about infinity are used) to give computable bounds for approximation to algebraic functions. Schmidt (to appear) shows that if $f(x, y) = c$ is a "Thue equation" of degree $d \geqslant 5$ whose coefficients are polynomials in t of degree $\leqslant \Delta$, then the rational solutions $x, y = u/w,\ v/w$ in reduced form have degree $u, v, w \leqslant 89\Delta$.

Another important result on approximation, but in a quite different direction, is contained in Thue's (1912b) paper. Thue shows that if a real number $\rho > 1$ has the property that $\rho^n = T_n + \varepsilon_n$ $(n = 1, 2, \ldots)$ where T_n is a rational integer and where $|\varepsilon_n| \leqslant c_0/k^n$ with $k > 1$, then ρ is algebraic. More then twenty five years later this theorem was rediscovered by Pisot (1938), and the above numbers ρ are now called Pisot-Vijayaraghavan numbers, although Thue has the priority. Thue points out that real algebraic integers $\rho > 1$ whose conjugates lie inside the unit circle have the property in question, and Pisot shows that every number ρ with the mentioned property is in fact an algebraic integer of this special type. Salem (1944) proved the set S of these numbers to be closed, and Siegel (1944) proved that the smallest element in S is the positive root $\theta = 1.324\ldots$ of $x^3 - x - 1$, and is isolated.

As was pointed out, Thue often used the box principle. Sometimes he derived results which had already been obtained by similar methods by Dirichlet and by Minkowski. Thue (1902) obtained the fact, nowadays called the Thue Remainder Theorem, that given $p > 1$ and given a, b which are coprime to p, there is an integer $q > 0$ such that

$$aq = \alpha p + h, \quad bq = \beta p + k \quad \text{with } 0 < h^2, k^2 < p;$$

in other words, the remainders of aq, bq after division by p are less than $\sqrt{p}$.

Thue dealt with various diophantine equations. For instance, he used his (1908c) result on Thue equations to show (1911a) that the equation $P(x, y) = Q(x, y)$ has only finitely many solutions if P, Q are forms of degrees $p > q$, $p > 2$, and to show (1917b) that $ax^2 + bx + c = dy^n$ has only finitely many solutions if $n > 2$ and if $b^2 - 4ac$ is no square. (More general results were proved by Siegel in (1926) and (1929).) It seems that Thue was forever interested in the Fermat conjecture. See his doodle on Fermat's equation on a front page. He gives (1917a) a discussion along unusual lines of the cubic Fermat equation. For the general Fermat equation Thue (1911b) applies a method of descent, which has, however, not led to further progress.

Fairly at the beginning of his career Thue established the geometric-number theoretic result that the densest packing of equal disks in the plane is the hexagonal lattice packing. He first gave a rather sketchy discussion in (1892), and a different and more rigorous (still open to some objections) argument in (1910a). The analogous problem in three dimensions is still open; it is conjectured that the densest packing of balls is a certain lattice packing.

See Fejes Tóth (1953) and C. A. Rogers (1964).

Thue's genius shows again in his papers on strings of symbols and on logic. Two of his papers on strings are devoted to the construction of strings of symbols belonging to a given finite set of symbols, which have no repetition. Call a string irreducible if no two equal symbols or two equal blocks of symbols are adjacent. Thue at the beginning of (1906) shows that if R is a finite irreducible string composed of the four symbols a, b, c, d and if R' is the string obtained from R by replacing a, b, c, d respectively by the blocks $A = adbacbc$, $B = abdacbc$, $C = abadcbc$, $D = abacdbc$, then R' is again irreducible. In this way, Thue is able to construct irreducible strings of arbitrary length. Later in the same paper Thue improves and extends this by constructing infinite irreducible strings of only three symbols a, b, c. In his later paper (1912a) Thue redefines irreducibility of an infinite string of n symbols to mean that two equal symbols or two equal blocks of symbols should be separated by at least $n - 2$ symbols between them. When $n = 2$ this is to mean that two equal blocks must not overlap, and for $n = 3$ it gives the irreducibility as defined earlier. Thue shows that for each $n > 1$ there is an irreducible doubly infinite string $\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots$ composed of n symbols. For some modern questions on avoidable patterns in strings of symbols see Bean, Ehrenfeucht and McNulty (to appear).

Of lasting importance is Thue's (1914) paper on transformations of strings of symbols by given rules. Suppose we are given a finite set of symbols $a, b, \ldots, k$, and we are given two series of finite blocks of these symbols, say

$$A_1, A_2, \ldots, A_n,$$
$$B_1, B_2, \ldots, B_n.$$

If P, Q are strings, Thue writes $P \sim Q$ if Q can be obtained from P by replacing a block A_i which occurs in P by B_i, or conversely by replacing a block B_i by A_i. Further he calls P, Q equivalent if there are strings $C_1, \ldots, C_l$ such that $P \sim C_1$, $C_1 \sim C_2, \ldots, C_{l-1} \sim C_l$, $C_l \sim Q$. The following question is posed by Thue and is called a big general question: To find a general method which allows one to decide, after a computable number of operations, whether two strings A and B are equivalent. In his earlier paper (1910b) he deals with an even more general question (involving a number of "concepts" and binary operations between elements of these concepts, and represented very nicely geometrically by "trees") and says: "Eine Lösung dieser Aufgabe im allgemeinsten Falle dürfte vielleicht mit unüberwindlichen Schwierigkeiten verbunden sein." (A solution of this problem in the most general case might pose unsurmountable difficulties.) Thue was right!

A list of symbols $a, b, \ldots, k$ together with "relations" $A_1 = B_1, \ldots, A_n = B_n$ where $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ are finite blocks of symbols (they are "words") is nowadays called a Thue System. Such a system defines a semigroup in terms of generators $a, b, \ldots, k$, and of defining relations. The question of Thue (now called the Word Problem for Semigroups) is to decide whether two given words are equal. Post (1947) proved the far reaching result that there exists a Thue System of only two letters a, b where this decision process is "recursively" unsolvable. Later Novikov (1955) and Boone (see Boone (1959)) showed that the word problem for groups is also unsolvable.

Questions of this type are now at the center of interest. Thue was far ahead of his time.

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Moduln und Ringe, by Friedrich Kasch, B. G. Teubner, Stuttgart, 1977, 328 pp., DM 52

Even though the concept of a ring was not formulated until the beginning of this century, rings had already been studied in the nineteenth century in the special cases of rings of algebraic integers, polynomial rings, power series rings and finite dimensional algebras over the real and complex numbers. Modules over rings are generalizations of vector spaces over fields, and were first studied by Dedekind and Kronecker over rings of algebraic integers and polynomials rings, in particular, in the special case of ideals.

The theory of rings and modules has in this century developed in various

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[1](#2) etc. signifies the position in the present collection of papers.

[2]We use the abbreviations K.V.S.S. for Kra. Vidensk, Selsk. Skrifter. I. Mat. Nat. Kl.