Of course it isn't too important but I've always thought that Pitt is responsible for the result that any $T: l_p \to l_q, p > q$, is compact. The authors ascribe this to Paley (without reference). But, enough of this!

The book is highly enjoyable reading for anyone and must reading for anyone interested in vector measures or the geometry of Banach spaces.

The book, like most first editions, has misprints. No one will have difficulty with "language operators" (p. 148) or "1converging" (p. 182) [when read in context] and serious readers will find the subscripts lost or interchanged in some of the displays.

Thus the only serious mistake is the misspelling of the reviewer's name (p. 253).

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Jordan pairs, by Ottmar Loos, Lecture Notes in Math., vol. 460, Springer-Verlag, Berlin and New York, 1975, xvi + 218 pp., $9.50.

Jordan pairs are a generalization of Jordan algebras and Jordan triple systems.[1] The archetypal example of a Jordan algebra is the hermitian $n \times n$ matrices $x^* = x$ (for $x^* = \bar{x}^t$ the conjugate transpose) under the product $U(x)y = xyx$, while an example of a Jordan triple system is the rectangular $n \times m$ matrices under $P(x)y = xy^*x$. Such Jordan systems have recently come to play important roles in algebra, geometry, and analysis. In particular, the exceptional Jordan algebra $H_3(K)$ of hermitian 3×3 matrices with entries from the Cayley numbers K has important connections with exceptional geometries, exceptional Lie groups, and exceptional Lie algebras.

Although the structure of finite-dimensional Jordan algebras is well known, the structure of Jordan triple systems is generally known only over algebraically closed fields. The main obstacle to attaining a complete theory for triple systems is the paucity of idempotents: most nonassociative structure theories lean heavily on Peirce decompositions relative to idempotents, and a general triple system may have few "idempotents" x with $P(x)x = x$. For example, the triple system obtained from the real numbers via $P(x)y = -xyx$ has no nonzero idempotents at all. However, a well-behaved triple system does have many pairs of elements (x, y) such that $P(x)y = x$, $P(y)x = y$ (in the above example, for any $x \neq 0$ we may take $y = -x^{-1}$). Such a pair furnishes a pair of simultaneous Peirce-like decompositions of the space, which could provide useful structural information if the two didn't keep getting tangled up in each other.

Even in Jordan algebras, many concepts involve a pair of elements (x, y). Frequently this takes the form of x having a certain property, such as idempotence ($x^2 = x$) or quasi-invertibility (invertibility of $1 - x$), in the y-homotope; this roughly corresponds to the element xy having that particular property, and so serves as a substitute for the associative product xy which doesn't exist within the Jordan structure. (The *y-homotope* of an associative

[1]For a quick background survey of these systems see the article, *Jordan algebras and their applications* in this issue.

algebra has twisted product $x \cdot_y z = xyz$, the y-homotope of a Jordan algebra has product $U^{(y)}(x)z = U(x)U(y)z \cong xyzyx = x \cdot_y z \cdot_y x)$.

An *isotope* is a homotope by an invertible element y; passing to an isotope roughly corresponds to changing the unit element from 1 to y^{-1}. It is often natural to treat a Jordan algebra together with all its isotopes as a single algebraic system: many properties are independent of isotopy, not dependent upon an artificial singling-out of a unit element. The *structure group*, consisting of the autotopies (isomorphisms of isotopes), is an important algebraic group which appears more naturally than the automorphism group, consisting of those autotopies fixing the disinguished unit element. Similarly the Lie algebra of the structure group (the *structure algebra*, consisting of "diffeotopies") often arises more naturally than the Lie algebra of derivations (those diffeotopies killing the unit element). The structure group of the exceptional Jordan algebra $H_3(K)$ is a Lie group of type E_6 while the automorphism group is of type F_4; the structure and derivation algebras are Lie algebras of types E_6 and F_4. The structure group really consists of *pairs* $(g, g^{\#})$ of invertible linear transformations with $g(U(x)y) = U(g(x))g^{\#}(y)$ (whereas automorphisms have $g(U(x)y) = U(g(x))g(y))$, which suggests we think of the elements x and y in the product $U(x)y$ as belonging to different spaces, with g acting on the x's and $g^{\#}$ on the y's.

Such paired systems first arose in M. Koecher's work on Lie algebras. In a well-behaved Lie algebra with graded decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$ (where the degree of the product is the sum of the degrees of the factors, $[L_i, L_j] \subset L_{i+j}$), the pair of spaces (L_1, L_{-1}) naturally carries Jordan-like products $Q(x_1, y_1)u_{-1} = [[x_1, u_{-1}], y_1] \in L_1$ and $Q(u_{-1}, v_{-1})x_1 = [[u_{-1}, x_1], v_{-1}] \in L_{-1}$, where L_0 acts as derivations of these induced products. (Note that there is no natural product on L_1 or L_{-1} themselves, since $[L_1, L_1] \subset L_2 = 0$ and $[L_{-1}, L_{-1}] \subset L_{-2} = 0$.) This Jordan structure can be used to "coordinatize" the Lie algebra. Only when L carries an automorphism exchanging L_1 and L_{-1} can both be identified with a single Jordan triple system, and only when there is an inner automorphism exchanging L_1 and L_{-1} is the coordinate system J a unital Jordan algebra. Thus it is the Jordan pair which appears in the general situation. This observation was implicit in the work of M. Koecher and explicit in the work of K. Meyberg, who coined the term "verbundene Paare" but concentrated his attention on the triple system case. O. Loos has also shown how Jordan pairs arise naturally as "coordinates" for algebraic groups with decomposition $G \approx G_1G_0G_{-1}$ into parabolic subgroups.

1. Jordan pairs, triple systems, and algebras. The algebraic structure which emerges from these examples is a pair of spaces (V^+, V^-) which act on each other (but not on themselves) like Jordan triple systems: a *Jordan pair* has products $Q(x_\varepsilon)y_{-\varepsilon} \in V^\varepsilon$ ($\varepsilon = \pm 1$) and derived products $D(x_\varepsilon, y_{-\varepsilon})z_\varepsilon = Q(x_\varepsilon, z_\varepsilon)y_{-\varepsilon} \in V^\varepsilon$ which satisfy the usual 3 Jordan identities (1) $Q(x_\varepsilon)D(y_{-\varepsilon}, x_\varepsilon) = D(x_\varepsilon, y_{-\varepsilon})Q(x_\varepsilon)$, (2) $D(Q(x_\varepsilon)y_{-\varepsilon}, y_{-\varepsilon}) = D(x_\varepsilon, Q(y_{-\varepsilon})x_\varepsilon)$, (3) $Q(Q(x_\varepsilon)y_{-\varepsilon}) = Q(x_\varepsilon)Q(y_{-\varepsilon})Q(x_\varepsilon)$. In case V^+ is $n \times m$ matrices and V^- is $m \times n$ matrices with $Q(x_\varepsilon)y_{-\varepsilon} = xyx$, these amount to $x(yxz + zxy)x = xy(xzx) + (xzx)yx$, $(xyx)yz + zy(xyx) =$

$x(yxy)z + z(yxy)x$, $(xyx)z(xyx) = x(y(xzx)y)x$.

We can build such a Jordan pair by "doubling" a unital Jordan algebra J: set $V^+ = V^- = J$ and $Q(x_\varepsilon)y_{-\varepsilon} = U(x)y$. These pairs contain elements v_ε (such as the unit element 1) for which "multiplication by v" $Q(v_\varepsilon)$ is a bijection $V^{-\varepsilon} \to V^\varepsilon$. Conversely, every Jordan pair with such *invertible* elements can be obtained in this manner, so Jordan pairs with invertible elements are equivalent to unital Jordan algebras up to isotopy. Under this equivalence the automorphism group and derivation algebra of the Jordan pair correspond to the structure group and structure algebra of the Jordan algebra. It is an open question whether one may always somehow "adjoin a unit" to a Jordan pair or triple system and thereby reduce it to a Jordan algebra.

Similarly we may double a Jordan triple system T to obtain a Jordan pair (T, T) via $Q(x_\varepsilon)y_{-\varepsilon} = P(x)y$. Such pairs have canonical exchange involution $(x, y) \to (y, x)$, and the category of Jordan triple systems is equivalent to the category of Jordan pairs with involution.

On the other hand, Jordan pairs may be thought of as special kinds of triple systems. A *polarized* Jordan triple system is a Jordan triple system together with a "symplectic" decomposition $T = T^+ \oplus T^-$ such that $P(T^\varepsilon)T^{-\varepsilon} \subset T^\varepsilon$, and $P(T^\varepsilon)T^\varepsilon = P(T^\varepsilon, T^{-\varepsilon})T^\varepsilon = 0$. We have inverse functors $T^+ \oplus T^- \leftrightarrow (T^+, T^-)$ between the category of polarized Jordan triple systems and the category of Jordan pairs.

2. Inner automorphisms and Peirce decompositions. The lack of a unit element in Jordan pairs causes a shift in emphasis towards concepts which provide an effective replacement. In Jordan theory an important role is played by the multiplication operator $U(x)$ of the element x (reducing to $z \to xzx$ in associative algebras); in the theory of Jordan pairs a corresponding role is played by the multiplication operator $B(x,y) = I - D(x,y) + Q(x)Q(y)$ of the pair (x,y) (reducing to $z \to (1 - xy)z(1 - yx)$ in the associative case). The quasi-inverse plays the role in Jordan pairs that the inverse does in Jordan algebras. O. Kühn has shown how the theory of Jordan pairs can be derived from quasi-inversion as the basic algebraic operation (just as T. A. Springer has derived Jordan algebras from inversion). The pair (x,y) is quasi-invertible iff the operator $B(x,y)$ is invertible (just as x is invertible iff $U(x)$ is invertible), in which case we obtain an inner automorphism $\beta(x,y) = (B(x,y), B(y,x)^{-1})$. Even in the case of Jordan algebras these $B(x,y)$'s seem more natural generators of the *inner* structure group than the $U(x)$'s. (Over an infinite base field both generate the same group of linear transformations, using $U(x) = B(1 - x, 1)$ and $B(x,y) = U(x)U(x^{-1} - y)$ and a Zariski density argument.) For example, the infinitesimal versions of the $B(x,y)$'s are the generators $D(x,y)$ of the *inner* structure algebra. The B's are called *Bergman operators* because the Bergman kernel function of a bounded symmetric domain is given by $k(x,y) = \det B(x,y)^{-1}$.

In a Jordan triple system one can only define odd powers of an element. In a Jordan pair one can define arbitrary powers, but only for a pair of elements (x,y) (where $x^{(n,y)}$ denotes the nth power of x in the y-homotope). These two notions of power are related: when (x,y) is considered as an element of the polarized triple system corresponding to the pair V, the odd triple powers are $(x,y)^{2n-1} = (x^{(n,y)}, y^{(n,x)})$. In this context, an *idempotent* is a pair (x,y) with $(x,y)^3 = (x,y)$, i.e. $x^{(2,y)} = Q(x)y = x$ and $y^{(2,x)} = Q(y)x = y$.

One of the key aspects of the theory of Jordan pairs is the broadness of this concept of idempotent. If $e^2 = e$ is idempotent in a Jordan algebra, or more generally $e^3 = e$ is tripotent in a Jordan triple system T, then (e,e) is idempotent in the Jordan pair (T, T), but in a well-behaved Jordan pair *every* element x may be completed to an idempotent (x,y). (In an associative matrix algebra, for example, by von Neumann regularity we can always find y' with $xy'x = x$, and a slight modification $y = y'xy'$ produces an element with $yxy = y$ as well as $xyx = y$.) By considering idempotents (x,y) where $x \neq y$ the theory admits a rich supply without losing any of the usual properties of idempotents.

As usual in nonassociative theories, Peirce decompositions $V = V_2 \oplus V_1 \oplus V_0$ relative to idempotents are key ingredients in unlocking the structure theory, reducing the general product on V to more concrete products between Peirce spaces V_i. (In the Peirce decomposition of the $(n+m) \times (n+m)$ hermitian matrices relative to the idempotent

$$e = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix},$$

V_2, V_1, V_0 consist respectively of all

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$$

for hermitian $n \times n$ matrices A, arbitrary $n \times m$ matrices B, and hermitian $m \times m$ matrices C.) The basic facts about Peirce decompositions follow cleanly via the 1-parameter group of inner automorphisms

$$\phi_+(t) = \beta(e_+, (1-t)e_-):$$

the homomorphism condition $\phi_\varepsilon(st) = \phi_\varepsilon(s)\phi_\varepsilon(t)$ yields the decomposition into eigenspaces V_2, V_1, V_0, while the automorphism condition $\phi_\varepsilon(t)(Q(x)y) = Q(\phi_\varepsilon(t)x)\phi_{-\varepsilon}(t^{-1})y$ yields the rules for multiplying these spaces.

3. Alternative pairs. The basic difference between Peirce decompositions of Jordan pairs or triple systems and those of Jordan algebras is that a well-behaved pair can have a "rectangular" Peirce decomposition $V = V_2 \oplus V_1$ with $V_0 = 0$ but $V_1 \neq 0$, whereas Peirce decompositions in an algebra are "square"–in a semisimple Jordan algebra $V_0 = 0$ forces $V_1 = 0$. The 3 basic examples of rectangular decompositions result from Jordan triple systems $T = T_2 \oplus T_1$ of matrices under $P(x)y = x(y^*x)$ as follows: (1) the rectangular matrices $T = M_{n,n+m}(D)$ with entries in an associative algebra D have $T_2 \cong M_{n,n}(D)$ and $T_1 \cong M_{n,m}(D)$, (2) alternating matrices of odd order $T = A_{2n+1}(k)$ have $T_2 \cong A_{2n}(k)$ and $T_1 \cong k^{2n}$, (3) the 1×2 matrices with entries in a Cayley algebra K have $T_2 \cong M_{1,1}(K)$ and $T_1 \cong K$.

A Peirce space V_1 carries a trilinear product $\langle x_\varepsilon y_{-\varepsilon} z_\varepsilon \rangle = D(D(x_\varepsilon, y_{-\varepsilon})e_\varepsilon, e_{-\varepsilon})z_\varepsilon$; in the above 3 examples this product $\langle xyz \rangle$ reduces to (1) xy^*z in $M_{n,m}(D)$, (2) $xy^*z + (Sy)x^*(S^{-1}z)$ in k^{2n} (S the standard alternating matrix of degree $2n$), (3) $(x\bar{y})z$ in K. In general, when $V_0 = 0$ this induced structure on V_1 may be axiomatically characterized as an *alternative pair* $A = (A^+, A^-)$ with product $\langle x_\varepsilon y_{-\varepsilon} z_\varepsilon \rangle \in A^\varepsilon$ obeying certain identities resembling those for alternative algebras; any such alternative pair has a "standard imbedding" as V_1 in a Jordan pair $V = V_2 \oplus V_1$. We may refine the structure of V_1 by considering further Peirce decompositions of it as alternative pair relative to orthogonal idempotents in V_1 (which of course are no longer orthogonal when considered in V). The numerous Peirce relations for alternative pairs suffice to show that a suitable simple alternative pair is one of the 3 basic types listed above. (This reviewer feels that it may be possible to avoid the complicated and asymmetric identities of alternative pairs by analyzing the Jordan pair directly using "collinear" rather than merely "orthogonal" families of idempotents.)

4. Structure theory. One of the triumphs of the theory of Jordan pairs is the complete structure theory for pairs with d.c.c. on inner ideals. Outside the case of finite-dimensional systems over algebraically closed fields there are too few idempotents to carry through a structure theory for Jordan triple systems. By passing from Jordan triple systems to Jordan pairs one is able to bypass these difficulties and push through the theory over an arbitrary ring of scalars. (To complete the classification of Jordan triples one needs to classify involutions in Jordan pairs, and this is still an open problem.)

The proper notion of "one-sided ideal" for a Jordan system is that of an *inner ideal*, a subspace $B^\varepsilon \subset V^\varepsilon$ closed under "inner multiplication" by V, $Q(B^\varepsilon)V^{-\varepsilon} \subset B^\varepsilon$. An important example is the *principal* inner ideal $B^\varepsilon = Q(x_\varepsilon)V^{-\varepsilon}$ determined by an element x. (In an associative matrix algebra A each left ideal has the form Af, each right ideal the form eA, and each inner ideal the form eAf for idempotents e, f; a principal inner ideal has the form xAx.) As in Jordan algebras, the natural finiteness restrictions are descending chain conditions on inner ideals.

Putting together the analysis of V_1 as alternative pair and the known structure theory of V_2 as unital Jordan algebra, the following basic structure theory is obtained: a "semisimple" Jordan pair with d.c.c. on inner ideals is a direct sum of simple pairs which are either (0) Jordan division pairs, (I) rectangular matrices $(M_{n,m}(D), M_{n,m}(D))$ for an associative division algebra D, (II) alternating matrices $(A_n(k), A_n(k))$ for an extension field k, (III) hermitian matrices $(H_n(D, D_0), H_n(D, D_0))$, (IV) "ample outer ideals" in the Jordan algebra $J(Q, c)$ of a nondegenerate quadratic form Q, (V) 1×2 Cayley matrices $(M_{1,2}(k), M_{1,2}(K))$, (VI) hermitian 3×3 Cayley matrices $(H_3(K), H_3(K))$. The latter two pairs, of dimension 16 and 27 over their centers, are the only pairs which are *exceptional* in the sense that they cannot be imbedded in associative systems. A slightly more general description is obtained for pairs with d.c.c. only on principal inner ideals, under the additional assumption of the existence of a maximal idempotent (equivalently, a.c.c. on principal inner ideals).

Here Types (0), (III), (IV), (VI) always contain invertible elements and come from Jordan algebras, as does (I) when $m = n$ and (II) when n is even, but (V) never does; it may be realized as a Peirce 1-space in (VI). Notice that all 6 Types of simple Jordan pairs with d.c.c. come from Jordan triple systems, and all can be imbedded in pairs with invertible elements: (I) has $M_{n,m}(D) \subset M_{n+m,n+m}(D)$, (II) has $A_{2m-1}(k) \subset A_{2m}(k)$, and (V) has $M_{1,2}(K) \subset H_3(K)$.

Summary. This book is a good place to find the most elegant modern methods in quadratic Jordan algebras and triple systems, though as it assumes basic results on the structure of alternative and Jordan algebras it is not completely self-contained.

The concepts of Jordan and alternative pairs introduced here for the first time generalize the previously developed theories of Jordan and alternative triple systems, and complete theories are obtained. Pairing sufficiently broadens the concept of idempotent to yield a structure theory for pairs with d.c.c. analogous to that obtained for Jordan and alternative algebras, but unattained for triple systems.

The theory of Jordan pairs not only recasts and completes the theory of Jordan triple systems, but provides a single algebraic system describing a Jordan algebra together with all its isotopes. The structure group receives its natural interpretation as the automorphism group of the associated Jordan pair. Jordan pairs arise naturally and spontaneously in Lie theory and in algebraic groups.

The theory of alternative pairs and triple systems is really an abstract treatment of certain Peirce 1-spaces in Jordan pairs and triple systems, and is not so intrinsically important.

Just as the passage from linear to quadratic Jordan algebras (or algebraic systems whose basic operation is inversion) requires a readjustment of habits of thinking, so the passage from Jordan triple systems to Jordan pairs requires new perspectives. Those who prefer to view a Z_2-graded algebra as a sum $A = A_e \oplus A_o$ of even and odd pieces rather than a union $A = (A_e, A_o)$, may use the equivalent concept of polarized Jordan triple system. Nevertheless there are real advantages to be gained from clearly separating a pair into distinct pieces. Those interested in nonassociative algebras and their applications would do well to gain facility in the language of pairs, and the present book is an excellent place to learn.

KEVIN MCCRIMMON