whose original proof was very difficult (and for which still a new, quite simple proof was found recently by Hörmander and Melin). Goodman's proof grows out naturally from his fundamental guiding idea of approximating one algebraic structure by another which he uses fruitfully at other points too.

The notes are mainly oriented towards the Rothschild-Stein theory of hypoellipticity and they contain an account of singular integral theory in a rather general setting well adapted to this application. But there are also two long sections about the applications to intertwining operators and to the Cauchy-Szegö integral. These contain clear detailed explanations of the original problems and their connections with other things. A particularly attractive feature is the inclusion in the section on the Cauchy-Szegö integral of an account of the work of R. D. Ogden and S. Vági which illuminates the problem from the side of harmonic analysis on H_n. There is also an interesting appendix on generalized Jonquières groups, and there is a good bibliography.

This set of notes, which could actually be called a book, is indispensable to anyone seriously interested in this promising new subject.

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Stochastic processes, by John Lamperti, Applied Mathematical Sciences, vol. 23, Springer-Verlag, New York, Heidelberg, Berlin, 1977, xiv + 266 pp., $9.80.

Ever since the appearance of J. L. Doob's 1953 book of that name, *Stochastic processes* has been a term to conjure up visions of elaborate mathematics applicable to studying the passage of time in random phenomena. It has been aptly remarked (I believe by Professor Lawrence Marcus) that the real universe is *either* a mechanical system of infinite dimension, *or* a stochastic process. If we admit the presence of random elements then only the second alternative is possible. In a topic of this breadth, however, it is inevitable that one does not make headway with a frontal approach but only by the maxim of "divide and conquer". One postulates various special properties which lend themselves to mathematical development, but one leaves the question of their universal applicability to others (presumably, to philosophers and theologians). It is perfectly sufficient that the results be interesting mathematically, and that they apply (to a sufficient degree) within very restricted areas of validity.

In writing a general text on stochastic processes, one is thus confronted at the outset with a dilemma. On the one hand, since a stochastic process is simply a family of random variables X_t, $t \in T$, on a probability space (Ω, F, P), there is little or nothing to be said about the subject as a whole. On the other hand, as soon as special further properties are assumed, the subject divides into domains which are rather far apart, both physically and mathematically, according to the differing natures of those assumptions. The situation is not unlike what one would encounter in biology if asked to write on the topic of "habitats". It is first of all necessary to specify what creatures are to be the inhabitants, and this makes a vast difference in the results!

Luckily, however, with stochastic processes the various assumptions almost never contradict each other except in very obvious ways, so that one can afterwards combine the various inhabitants without much jeopardy. Thus, for example, a Markov process of course cannot both have continuous paths and have jumps. But it is often quite easy to understand the sum of a continuous and a pure-jump process in terms of the separate components. The main conceptual difficulty of Markov processes with jumps arises rather from the fact that the general process (without discontinuities of the second kind) cannot be decomposed into a sum of continuous and pure-jump components. This problem, however, is outside the scope of the book under review.

A more pervasive distinction is that between assumptions of stationarity, on the one hand, and the approach to processes through conditional probability. Unlike most other treatments, the present one begins with stationary processes. The practical meaning (somewhat paradoxically) is that "This sort of process often can describe a physical system which is in a steady state, but which continually undergoes random fluctuations." Mathematically, of course, it simply means that the structure is invariant under the additive group acting on the time coordinates. However, the novelty of the present work is even larger, since wide-sense stationarity precedes strict-sense stationarity. In wide-sense stationarity, only the moments up to order 2 are assumed invariant. This means in effect that the process X_t, $t \in T$, is treated as a curve in Hilbert space, associated with a group U_t of unitary operators by the equations $X_t = U_t X_0$. Since only these moments may be used in deriving results, all operations such as limits, derivatives, integrals, etc., are carried out in the Hilbert space sense. The famous interpolation and prediction theory of A. Kolmogorov, N. Wiener, and others, falls within this framework, which occupies Chapters 2 and 3 of the present work. The actual content consists very largely of "old chestnuts", but it is expertly done and pleasingly organized.

Nevertheless, one should not overlook the uncomfortable question of how a theory can accomplish so much which does not distinguish between the two processes $X_n = \int_{-\pi}^{\pi} e^{in\lambda} dB_\lambda$ and $Y_n = \sum e^{in\lambda_j}$, where B_λ is a Brownian motion and $\{\lambda_j\}$ are the jump times in $(-\pi, \pi)$ of a Poisson process with unit intensity. Here, of course, the entire distinction rests in strict-sense properties which are not of the wide sense. Accordingly, it is quite logical to let the more general and opaque wide-sense theory precede the more detailed and precise strict-sense theory, as Lamperti has done. We can only regret that the latter is allotted just 21 pages, permitting only a few standard examples and the usual short proof of the ergodic theorem. Thus the quotation of the text that the ergodic theorem is "a theorem looking for a theory" almost seems to be a self-fulfilling prophecy.

The other approach to processes, namely through conditional probability, occupies the second half of the book. The heuristic, probabilistic meaning of this approach is roughly that the process is regarded as a kind of transient evolution from the given states to the yet unrealized states, according to fixed, conditional probability laws. The simplest case is that of Markov processes, where application of the laws requires only the present value of the process. The laws are then expressed by what is called the transition function. This, together with the initial distribution, entirely determines the joint distributions of the process. To proceed very far probabilistically with Markov processes, however, one requires a knowledge of martingale theory. This last corresponds to Markov processes in somewhat the same way that wide-sense stationarity corresponds to strict sense. Thus the martingale assumption, viz. that the process is centered at conditional expectations given the entire past, does not lead to a complete determination of the process, but only to certain general qualitative properties.

The author's strategy at this point, however, is opposite to that in the first half of the book, in the sense that martingale theory is postponed until the final chapter. Therefore, instead of beginning Markov processes with the probabilistic content, he leads into the subject through the analytic theory of the transition function "from a somewhat old-fashioned point of view." A rather general analytic discussion is first made of finite state Markov chains in both discrete and continuous time. Here the transition function is simply iteration or exponentiation of a finite matrix, and one can obtain the limiting behavior as $t \rightarrow \infty$ without much difficulty. The author does not strive for complete generality but rather emphasizes methods (at the cost of an extra positivity hypothesis). It is then easy to introduce the general state space in discrete time, and the compound Poisson processes in continuous time. Since all of this is only introductory to the following chapter entitled *The application of semigroup theory*, most of the hard questions and examples are carefully avoided.

In Chapter 7, operator semigroups are introduced as "the skeleton key which brings order out of all this chaos." There is, of course, on any measurable space a one-to-one correspondence between time-homogeneous Markov transition functions, and positive contraction semigroups of measures. But what is most useful about the semigroup approach is that the adjoint semigroup on the bounded measurable functions is usually much more tractable than the original one. In most cases of interest, one has a metric space, and the adjoint has the "Feller property" of transforming the bounded continuous functions into themselves. It is then relatively innocuous to assume strong continuity on the subspace of uniformly continuous functions, in the uniform norm. With the additional hypothesis that the metric space is compact, we reach the basic setting of Lamperti's text. A nice form of the Hille-Yosida Theorem then provides a characterization of the infinitesimal generators A of the semigroups, chiefly by the property that $Af(x_0) \leqslant 0$ at a maximum x_0. This can be applied to identify the generators and transition functions of most of the familiar diffusion processes, for which, of course, A is an unbounded differential operator.

By and large, all of this is classical analysis with a smattering of probabilistic content. The semigroups are determined, but what about the behavior of the processes? More damaging than this question (which can be effectively postponed) is the related one of the probabilistic meaning and role of the analytical hypotheses which have been introduced. When, in the final chapters, "We now turn to this neglected side of our subject," one cannot escape the feeling that to a degree the die is already cast. It is too late to effectively study complete intrinsic classes of processes probabilistically, since this would clash with the restrictive nature of the analytical assumptions already made. The most that one can do is to examine the probabilistic implications of these hypotheses. In this way, the emphasis on semigroup theory tends to become a kind of analytical straight jacket for subsequent developments, as seen from a more probabilistic standpoint. Thus we would characterize the last 80 pages, not as a "survey of the mathematical theory," but rather as a succinct introduction to the probabilistic behavior of Markov processes.

It is convenient here to distinguish qualitative from quantitative properties of a process. For example, right-continuity of paths is a qualitative property, while a first passage time distribution is quantitative (in the probabilistic sense). As a rule, qualitative problems arise only in continuous time. This is first apparent in the observation that a continuous time transition function does not really determine a process. Since there is latitude up to sets of probability 0 for each t, these sets can accumulate to be nonmeasurable or of positive probability. Consequently, there is a real basic difference between continuous and discrete-time processes, which cannot be obviated by use of discrete time approximations. It requires an entire chapter to show, for example, that for transition functions satisfying the analytical hypotheses noted above there always exist processes whose paths are right-continuous, with left limits for $t > 0$. But even this does not really explain "why". Analogous remarks apply to the result that the paths may be taken to be continuous if the generator A is of local character on a sufficiently large domain.

In the penultimate Chapter 9, entitled *Strong Markov processes*, we begin to understand how these difficulties arose. The transition function provides information only about constant times. But with a continuous time parameter, even the simplest random times do not readily reduce to constant ones. In order to commence the solution of more quantitative problems, one must first define a suitably wide class of random "stopping times," at which the transition function continues to describe the conditional law of passage from past to future. Fairly delicate concepts and arguments are needed to reach the conclusion that passage times to open and closed sets indeed have this property. The next step is then to introduce the characteristic operators, which are analogous to the infinitesimal generators but with the roles of time and space more or less interchanged. This leads naturally toward the relative theory of processes on subdomains, and the introduction of boundaries and boundary conditions.

Except for the final short chapter on martingales, the work concludes with two applications: a characterization of Brownian motion based on symmetry, and a probabilistic solution of the Dirichlet problem for bounded, convex domains. The latter is intended to give "a glimpse . . . into the extensive and rewarding new topic of probabilistic potential theory," thus making it clear that the book is intended to be an introduction as well as a survey.

Despite the uniformly high standard and quality of the work, we must conclude with a couple of mildly critical remarks. First, it seems regrettable that there are no applications of the machinery of strong Markov processes to the computation of actual probabilities. Granting that the emphasis is on processes rather than probabilities, it nevertheless seems premature to go on to applications to analysis without first computing even one real probability distribution, be it for a passage time, a hitting probability, an occupation time, or some more involved functional. Secondly, the overall tone of the work is already set in the preface as follows: "The great day of the dedicated solitary researcher is over, if indeed it ever existed. . . . In their stead, concern for the human consequences of scientific and technological achievement must become part of our working lives, . . . Only through organized collective action can this be achieved." This being so, it is easy to imagine why the methods and ideas of a generation of researchers should be presented here in a condensed and transparently clear form, with no suggestion of the effort that must have gone into developing them. Professor Lamperti has indeed done a highly praiseworthy job in providing us with a careful and painless review of stochastic processes. For some readers, however, the work may be a trifle unoriginal. A few more novel calculations, descriptive generalities, or even loose ends, might have alleviated the collective mentality and given the reader more to remember.

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Vector measures, by J. Diestel and J. J. Uhl, Jr., Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977, xiii + 322 pp., $35.60.

I am an avid reader of the mystery novels of John Dickson Carr and the Poirot stories of Agatha Christie. I was led to these authors by a keen earlier interest in the works of Edgar Allen Poe and the Sherlock Holmes Stories of Sir Arthur Conan Doyle. Thus, in good faith, I cannot say that this book under review is the most entertaining book I've read; however, I can say that it is the most entertaining mathematics book I've ever read (including a famous measure theory book much enjoyed in my wasted youth). Indeed this serious, but sometimes irreverent, romp through vector measures can be enjoyed even by those misguided souls with a strong dislike for vector valued integration and the geometry of Banach spaces.

I will go so far as to say that the introduction alone is worth the (exorbitant?) price of the book: " . . . shortly after 1936, Dunford was able to recognize the Dunford-Morse theorem and the Clarkson theorem as genuine Radon-Nikodym theorems for the Bochner integral. This was the first Radon-Nikodym theorem for vector measures on abstract measure spaces."

"B. J. Pettis, in 1938, made his contribution to the Orlicz-Pettis theorem for the purpose of proving that weakly countably additive vector measures are norm countably additive."

" . . . Dunford and Pettis, in 1940, built on their earlier work to represent weakly compact operators on L_1 and the general operator from L_1 to a separable dual space by means of a Bochner integral. By means of their integral representation they were able to prove that L_1 has the property now known as the Dunford-Pettis property."

"Then came the war! By the end of the war, the love affair between vector measure theory and Banach space theory had cooled. They began to drift down separate paths. Neither prospered. Much of Banach space theory