introduces them in §4 in connection with the "product formula" for multiplicities. He also proves the basic "multiplicity one" theorem for $L^2(S_{\mathbf{Q}} \setminus S_{\mathbf{A}})$ for his examples. However, he does not sufficiently exploit them in this reviewer's opinion. Particularly, in his proof of the product formula he uses them only in a roundabout cumbersome way, while they could be brought to bear directly and incisively. (A technical point: in the discussion of the product formula for the hyperbolic case, class number one should be assumed, but the author does not do so explicitly.) Also the general conditions for multiplicity one are not discussed; it can fail if certain first Galois cohomology classes are nontrivial. See [Co] for an example. Also the adèlic viewpoint might have simplified and clarified §6, especially the role and structure of the oscillator or Weil representation for finite rings.

Such, in brief, is the content of the book. What of the form? The book is very carefully written (not to say proofread–there are numerous typos of a harmless sort, some amusing). The author does his best to communicate the subject as he understands it. He provides many tactical and motivational asides, so even though the material is fairly technical, the reading is usually tolerable. The author's care and his down-to-earth approach will be appreciated by those wishing to learn the subject. In the first part, he really does a very good job of presenting a lot of material with minimal prerequisites. On the negative side, one feels sometimes the forest is being lost for the trees, as for example in the failure to distinguish before §10 between true solvmanifolds and the much simpler but important subclass of nilmanifolds. It would be nice to have a redo of the same material at a much higher level of sophistication. (This is perhaps too much to ask of one author.) Also somewhat dampening is the author's very pessimistic attitude toward his subject, the more so because in several spots where the author paints in his darkest palette, as in §6 and §8, some technical improvements could make the picture rosier. However as a painstaking introduction to a subject that deserves more attention and offers considerable potential for development, and for its attractive examples, (not to mention the fact that it is the only place you can read about a considerable portion of its content) the book is a valuable contribution.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, July 1978

Nilpotent Lie groups: Structure and applications to analysis, by Roe W. Goodman, Lecture Notes in Math., vol. 562, Springer-Verlag, Berlin and New York, 1976, x + 209 pp., $10.20.

In this review a *nilpotent group* will always mean a connected and simply connected Lie group. This is the same thing as a real vector space V which has a group structure such that 0 is the neutral element and xy^{-1} is a V-valued polynomial on $V \times V$; so trivially, any V with vector addition is a nilpotent group. The simplest nontrivial example is the "Heisenberg group" H_n, which can be regarded as a group of affine transformation on $\mathbf{C}^{n+1} = \mathbf{C} \times \mathbf{C}^n$; here $V = \mathbf{R} \times \mathbf{C}^n$ and the element $(a, c) \in \mathbf{R} \times \mathbf{C}^n$ acts by

$$(z, \zeta) \to \left(z + a + 2i \sum \bar{c}_j\zeta_j + i \sum |c_j|^2, \zeta + c\right).$$

Further examples are the group of strict upper triangular matrices and its subgroups which can be quite complicated.

Much is known about nilpotent groups. Their unitary representations have a very satisfactory description and their Plancherel formula is known; analysis on the quotient of the group by a discrete subgroup has made important progress throwing, in particular, some new light on the theory of theta-functions. These results, however, are not the subject of the present book; it is concerned almost exclusively with another line of development which started in 1968/1969 with two completely independent investigations.

The more elementary one of these was the work of S. Vági and the reviewer on the Cauchy-Szegö integral of the complex unit ball. To make things simpler, we transform the ball of $\mathbf{C}^{n+1}$by a fractional linear transformation to the "halfplane" given by

$$D = \left\{(z, \zeta) \in \mathbf{C} \times \mathbf{C}^n | \operatorname{Im} z - \sum |\zeta_j|^2 > 0\right\}. \tag{1}$$

Now the Cauchy-Szegö integral assigns a homomorphic function to any L^2-function f on the boundary B of D; denoting the limit of this on B by Pf, P is an orthogonal projection of $L^2(B)$. The question is whether P is a bounded operator on other L^p-spaces too (for $n = 0$ this is true for $1 < p < \infty$ by a classical theorem of M. Riesz) and whether P preserves Lipschitz spaces.

H_n acts on D and on B, in fact it can be identified with B. Our integral then becomes a convolution on H_n with respect to the noncommutative group structure, and P becomes a limit of convolution operators analogous to a Cauchy principal value, but with the Euclidean distance replaced by a "gauge" on H_n which is homogeneous under the automorphisms $(a, c) \to (t^2a, tc)$, $t > 0$. What was needed, therefore, for an answer to our question was an analogue of a basic Calderón-Zygmund theorem on singular integrals for H_n instead of $\mathbf{R}^n$. Such an analogue was proved for any nilpotent group with a homogeneous gauge along classical lines, only one of the essential arguments had to be replaced by a new one.

Exactly at the same time A. W. Knapp and E. M. Stein were studying the principal series of unitary representations of a semisimple Lie group G. These representations act on certain Hilbert spaces of functions on a nilpotent subgroup; there are several of them that are equivalent, and the study of the intertwining operators of the equivalent ones leads to important results about the irreducibility question. This study can to a large extent be reduced to the case where G has real rank one. Knapp and Stein found that in this case the intertwining operators were singular integrals on a nilpotent group with homogeneous gauge, exactly in the sense described above. In this case, however, the problem was the L^2 boundedness of the operators, which in the case of the Cauchy-Szegö integral was trivial. The usual method of the classical theory–applying the Fourier transform–was of no use here; Knapp and Stein settled the question by developing another method, suggested earlier in the $\mathbf{R}^n$-case by M. Cotlar.

At this point all the basic facts about singular integral operators were generalized to nilpotent groups with homogeneous gauge. There had been still further simultaneous and independent work in a similar direction: The L^p-boundedness theorem of singular integrals was extended by N. Riviere and in joint work by R. Coifman and M. de Guzmán to rather general locally compact groups or homogeneous spaces rigged with some further structure. But it was the nilpotent group case that became the starting point of further important developments, much in the same way as the classical basic results on singular integrals were the starting point of the theory with variable kernel, i.e., pseudo-differential operators. Analogously to the use of the classical theory to prove regularity theorems for elliptic problems the new theory became applicable to hypoelliptic problems.

These developments were realized in two major steps. The first was the work of G. B. Folland and E. M. Stein on the $\bar{\partial}_b$-problem on the boundary of a strongly pseudo-convex domain in $\mathbf{C}^n$. This problem can be treated with the aid of the associated Kohn Laplacian; for p-forms with $p \neq 0, n$ this gives a hypoelliptic problem. In the case of the domain (1) one works on H_n and finds an explicit fundamental solution. In the general case one constructs a parametrix by approximating the boundary by the boundary B of (1). Finally one gets sharp regularity theorems by using modified Sobolev spaces which involve differentiations of different order in different directions.

The second step was the work of L. Rothschild and E. M. Stein. Given vector fields $X_0, X_1, \ldots, X_n$ on a manifold M such that their successive brackets span the tangent space at every point, it is a well-known theorem of Hörmander that $X_0 + \sum X_j^2$ is hypoelliptic; this theorem is reproved and the accompanying Sobolev estimates are sharpened by the new methods. The group used here is the free nilpotent group N_r of step r, where r is the minimum degree of brackets needed to span the tangent spaces. A "lifting theorem" is proved first, i.e. additional variables besides those of M are introduced and the fields X_j extended so as to eliminate all the nongeneric relations up to order r among their brackets. (At the end one "descends" by integrating out the new variables.) Next, a mapping has to be constructed between the extended manifold and N_r with good enough properties so that results about N_r can be transferred to the manifold. At that point one can use the basic facts and some results of Folland about hypoellipticity on the group.

All this takes much work, but the basic ideas are simple, and one gets important new insights into the nature of hypoellipticity.

The lecture notes of R. W. Goodman succeed remarkably well in organizing and coherently expounding much of this material. They also do a good deal more by using some original new methods and clearing up some important points. An essential novelty is a new proof of the lifting theorem whose original proof was very difficult (and for which still a new, quite simple proof was found recently by Hörmander and Melin). Goodman's proof grows out naturally from his fundamental guiding idea of approximating one algebraic structure by another which he uses fruitfully at other points too.

The notes are mainly oriented towards the Rothschild-Stein theory of hypoellipticity and they contain an account of singular integral theory in a rather general setting well adapted to this application. But there are also two long sections about the applications to intertwining operators and to the Cauchy-Szegö integral. These contain clear detailed explanations of the original problems and their connections with other things. A particularly attractive feature is the inclusion in the section on the Cauchy-Szegö integral of an account of the work of R. D. Ogden and S. Vági which illuminates the problem from the side of harmonic analysis on H_n. There is also an interesting appendix on generalized Jonquières groups, and there is a good bibliography.

This set of notes, which could actually be called a book, is indispensable to anyone seriously interested in this promising new subject.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, July 1978

Stochastic processes, by John Lamperti, Applied Mathematical Sciences, vol. 23, Springer-Verlag, New York, Heidelberg, Berlin, 1977, xiv + 266 pp., $9.80.

Ever since the appearance of J. L. Doob's 1953 book of that name, *Stochastic processes* has been a term to conjure up visions of elaborate mathematics applicable to studying the passage of time in random phenomena. It has been aptly remarked (I believe by Professor Lawrence Marcus) that the real universe is *either* a mechanical system of infinite dimension, *or* a stochastic process. If we admit the presence of random elements then only the second alternative is possible. In a topic of this breadth, however, it is inevitable that one does not make headway with a frontal approach but only by the maxim of "divide and conquer". One postulates various special properties which lend themselves to mathematical development, but one leaves the question of their universal applicability to others (presumably, to philosophers and theologians). It is perfectly sufficient that the results be interesting mathematically, and that they apply (to a sufficient degree) within very restricted areas of validity.

In writing a general text on stochastic processes, one is thus confronted at the outset with a dilemma. On the one hand, since a stochastic process is simply a family of random variables X_t, $t \in T$, on a probability space (Ω, F, P), there is little or nothing to be said about the subject as a whole. On the other hand, as soon as special further properties are assumed, the subject divides into domains which are rather far apart, both physically and mathematically, according to the differing natures of those assumptions. The situation is not unlike what one would encounter in biology if asked to write on the topic of "habitats". It is first of all necessary to specify what creatures are to be the inhabitants, and this makes a vast difference in the results!