corresponds to a Lie subgroup, strikes me as one of those necessary but dull routines which are found in the elements of almost every mathematical subject. The classification of simple Lie algebras, on the other hand, is a stunning example of how beautiful linear algebra can be. The inclusion of the classification theorem at the end is a delicacy for the medicine the student has been forced to swallow; when so much of the proof is missing, however, the student is unable to recognize the treat for what it is.

There seem to be only a few misprints; e.g., the reference to A.3.9 on p. 146 should be A.1.6, and "notamment" is misspelt on p. 152. None of them should cause trouble. Each of the six chapters ends with historical notes and some exercises; many of the latter extend remarks and fill gaps in the text

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Mathematical theory of economic dynamics and equilibria, by V. L. Makarov and A. M. Rubinov, Springer-Verlag, New York, Heidelberg, Berlin, 1977, xv + 252 pp. $32.10.

In 1932 von Neumann presented a paper at a mathematics seminar at Princeton which was published six years later at the request of K. Menger [5] and translated into English in 1945 [6]. The ideas which form the main theme of the book under review, especially the key concepts of "economic dynamics" and its relation to "equilibria", appeared here in explicit mathematical form, probably for the first time. The von Neumann paper is one of the two "primary sources" for this book (the other will be mentioned later), and as such seems a convenient starting point for this review.

Here is a paraphrase of what von Neumann did. He considered an economic world in which there are n goods and a system of production, a *technology*, which can be described in the following simple manner. A *process* consists of a pair (a, b) of nonnegative n-vectors $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ where the entries a_i and b_i represent the amounts of the ith good. The physical interpretation is that if the vector a is available in an *input* at some time t then the vector b can be obtained from it as an *output*, becoming available at time $t + 1$. Processes are assumed to be *positively homogeneous* so that if (a, b) is a process so also is $(\xi a, \xi b)$ for any $\xi \geqslant 0$. Further if (a, b) and (a', b') are processes it is assumed that they may be operated simultaneously so that $(a + a', b + b')$ is also a process. A technology T is simply a set of processes satisfying these two conditions, thus, a subset of $R^n \times R^n$ which forms a convex cone.[1]

Given a technology T one is interested in its behavior over time. A *trajectory* of T is a sequence (a_t, b_t) of processes in T such that

$$a_t \leqslant b_{t-1}. \tag{1}$$

The condition has the obvious meaning that the inputs of any period must be taken from the outputs of the previous period (generally one assumes that some initial input, a_0, is given). In von Neumann's paper one is interested in a very restricted class of trajectories, namely, *steady states*, that is trajectories of the form

$$(a_t, b_t) = \alpha^t(a, b) \tag{2}$$

where α is some positive number called the *growth factor* of the trajectory and (a, b) is some fixed process in T. Clearly (2) represents an economy which is growing uniformly at a constant rate.

So far our considerations have been entirely technological, concerned only with goods and their production. To bring in some "economics" one assumes that at each instant of time there is a price associated with each good, thus, a nonnegative n-vector $p_t = (p_{1t}, \ldots, p_{nt})$. An *economic trajectory* is then a sequence of triples $(a_t, b_t; p_t)$. The key notion of the paper and also of the book, and indeed of all of economic theory, is that of *economic equilibrium*. In the present framework it can be introduced in the following simple manner: the *value* of the vectors a_t and b_t are $p_t a_t$ and $p_{t+1} b_t$ respectively (where juxtaposition of vectors means scalar product) the *profit factor* ρ_t of (a_t, b_t) is $p_{t+1} b_t / p_t a_t$. An *equilibrium trajectory* is one in which for each t the process (a_t, b_t) maximizes ρ_t among all (a, b) in T. The economic meaning of this condition is straightforward. Among all the possible processes, producers will choose only those which yield maximum profits. (The reason for using the term "equilibrium" may not be so obvious to noneconomists. Roughly speaking, the idea is that if at any time the process being operated was not profit maximizing then something would tend to change; either producers would try switching to a more profitable process or competition among producers would bring about a change in prices.)

The von Neumann problem was precisely the following: Do there exist steady state equilibrium trajectories? The answer–under some further technical assumptions on the technology–turned out to be yes. Further it was shown that if the growth factor of goods is α then the equilibrium prices p_t have the growth factor $1/\alpha$. Further, the number α is unique (though the prices and processes need not be) and turns out to be the maximum possible growth factor of the technology. As to the proof of this elegant result the story is quite interesting. In the introduction to his paper von Neumann says (I quote from the 1945 translation) "The mathematical proof is possible only by means of a generalization of Brouwer's fixed-point Theorem, i.e. by the use of very fundamental topological facts.... The connection with topology

[1]In von Neumann's paper T was assumed to be *finitely generated* in that all processes were assumed to be positive linear combinations of a given finite set of processes. For the present exposition, however, we need not impose this restriction.

may be very surprising at first but the author thinks that it is natural in problems of this kind." The proof is long and quite difficult. Four years earlier, in 1928 von Neumann had proved his famous minimax theorem of game theory also by using the Brouwer Theorem, and, in fact, as is mentioned in a footnote, the minimax theorem can be obtained as a special case of the steady state equilibrium theorem. Von Neumann was wrong, however, in believing that one needed theorems of algebraic topology for these results. In the same year, 1938, a French economist J. Ville gave an elementary proof of the minimax theorem **[10]** which depended only on separating a pair of convex sets by a hyperplane. It has turned out, in fact, that convexity theory rather than algebraic topology is the appropriate tool not only for the von Neumann problem but for a wide range of similar economic problems. (There are, however, a whole circle of the economic problems whose solutions do require fixed point methods as will be noted shortly.)

It is time to relate some of the above to the book under review. After a chapter of mathematical preliminaries, *Theory of Point-Set Maps* (this would have been better translated *point to set maps* or *set-valued maps*) there is a chapter entitled *The Neumann-Gale Model* which is devoted to extending the von Neumann existence result. It was recognized by many writers that von Neumann's uniqueness result depends on a very strong and unnatural assumption on the technology. When this is removed it turns out that the models may have a finite number of distinct growth rates. The original work on this problem is due to Kemeny, Morgenstern and Thompson **[4]** but the results in this book are more general. The growth rates which occur are related to eigen-values of *super linear maps* (this is new material due to the authors).

Chapter 3 is entitled *Optimal Trajectories and their Characteristics*. The subject matter is roughly the following: Suppose a technology is to be operated for T periods and the objective is to maximize the amount of some good, say, guns or butter or some weighted combination of these at time T. More realistically, a five-year plan might have as its goal to maximize some pre-assigned function f of the goods in the economy at the end of the planning period. The function f would be increasing in the "desirable" goods but decreasing in undesirable ones such as air pollution. A T-period trajectory which achieves this maximum is said to be *optimal with respect to f*. The main result of the chapter is to show the relationship between optimal trajectories and those which have *characteristics*. This latter term is peculiar to this book but in the context of the von Neumann model a characteristic would simply be a price sequence (p_t) such that the sequence $(a_t, b_t; p_t)$ is an equilibrium trajectory. This chapter seems to me at once the most interesting and most irritating in the book. The thing which appeals to me is a slick way in which the authors handle time. In all other writings I know of, models use either discrete time and difference equations or continuous time and differential equations. The authors introduce a device which includes both. The time variable runs over an arbitrary subset E of real numbers. For each t in E there is a subset K_t of some finite dimensional space representing the goods vectors attainable at time t. Next, for any $t < t'$ in E there is a function $a_{t,t'}$ from K_t to subsets of $K_{t'}$, the meaning being that y is in $a_{t,t'}(x)$ if, given the vector x at time t, it is possible to produce in such a way as to have vector y available at time t'. The natural conditions of compatability now require

$$\text{for } t < t' < t',\ a_{t',t''} \circ a_{t,t'} = a_{t,t''}. \tag{3}$$

A *trajectory* starting from an initial point x_{t_0} in K_{t_0} is then a choice function $x_t \in K_t$ such that $a_{t,t'}(x_t) = x_{t'}$. This is a natural framework for setting up general initial value problems and the like. The subsequent development is quite elegant.

My complaint is with the second half of the chapter in which we are asked to work through more than fifty pages of fairly heavy mathematics concerning the existence of characteristics without even being told what they are, economically, or why we should be interested in them. Perhaps this is just poor exposition on the part of the authors. Another possibility is that the much needed motivational material on prices and profits would lead too far into politically sensitive areas. I know from conversations with one of the authors several years ago that there was a period in the USSR when even "equilibrium" was a dirty word. On the other hand, prices and profits are mentioned quite freely in other parts of the book–so the matter remains mysterious.

Chapter 4, entitled *Asymptotes of Optimal Trajectories* provides a nice synthesis of the material of the previous two chapters. The history is something like this. In 1958 Dorfman, Samuelson and Solow [3] put forth a proposition to the effect that any "long" optimal trajectory, regardless of the objective function f or initial stock x_0, will be close to the von Neumann maximum growth trajectory "most of the time." This has come to be known as the "turnpike property", the metaphor being that the von Neumann trajectory is like a super highway along which maximum speed is possible. The proposition must of course be phrased with care, and the proofs turn out to be quite involved. The mathematically interesting fact is that the techniques required to establish these results make heavy use of prices and equilibrium trajectories although the formulation of the problem does not involve these concepts at all.

Chapter 5 is entitled *Models of Economic Equilibrium* and represents a change of pace. There is almost no dynamics in this chapter. Instead it treats the classical general equilibrium theorem which has come to be known as the Arrow-Debreu Theorem [1], namely a theorem which guarantees the existence of prices which balance supply and demand. All known proofs of the theorem make use of fixed point methods of algebraic topology or something equivalent, and this is necessarily the case since the Brouwer Theorem is an easy consequence of the existence of equilibrium prices (this is the situation referred to earlier). At this point the authors commit a bibliographical howler. They write "The original Arrow-Debreu model was not related to game theory, in particular it was not related to Nash equilibria. Our exposition is in terms of n-person game theory", and later, "The idea of linking the Arrow-Debreu model to an n-person game was suggested by Volkonskiĭ and implemented by Makarov." In fact, the proof of equilibrium in [1] is precisely by formulating a generalized game and proving existence of a Nash equilibrium. Clearly the authors had never read the Arrow-Debreu paper, for Makarov's implementation of Volkonskii's suggestion is almost identical with the proof in [1]. How could such a thing have happened? It must be that the authors read the book of Debreu [2] which is and perhaps always will be the classical reference in this area, and assumed that the proof given there was the same as that of the original Arrow-Debreu paper. I don't mean to make too much of this point. I'm more disturbed by the fact that, knowing the existence of the elegant direct proof as given in [2] (it was also found independently by Nikaido [8]), the authors instead present the relatively cumbersome one via game theory. In general, it becomes clear in reading the book that one of the author's main objectives is to create an outlet for their own work, an objective which is often at cross purposes with the principles of good exposition. Another example of this is the last section of Chapter 3, entitled *Spectral Theory of Super Linear Maps* in which one runs into some fairly high brow functional analysis e.g. the Schauder fixed-point principle, which as far as I could see never led to any economic pay-off.

In the final chapter, *Models of Economic Dynamics with Explicit Consumption* the authors bring together in a rather nice way almost all of the ideas which have gone before. Returning to the trajectories (a_t, b_t), a model with consumption assumes that some part of the output b_t is distributed to "consumers" as a *consumption vector* c_{t+1}. The new input vector a_{t+1} then clearly satisfies $a_{t+1} = b_t - c_{t+1}$, and one is interested in choosing among all technologically feasible consumption sequences (c_t). One adopts some criterion as to when one consumption sequence is better than another and tries to find an optimal sequence and describe its properties. The pioneering work on this problem was done by Frank Ramsey in 1928 [9], the same year von Neumann proved the minimax theorem (this is the second primary source referred to earlier). Ramsey worked with a model having a very simple technology (there was only one good) and a natural optimality criterion, and by a most ingeneous argument came up with an explicit formula for determining the optimal rate of consumption. A more detailed description would require too much space, but I can't resist digressing to say something about Ramsey himself. He is the same Ramsey of Ramsey's Theorem and the Ramsey Numbers, which must surely be one of the most influential developments in combinatorics of this century. Indeed, as I write this I note that at the winter meeting of the American Mathematical Society there will be two special sessions devoted to *Ramsey Theory and its Ramifications* at which 13 papers having Ramsey's name in the title will be presented. Besides [9] Ramsey wrote only one other paper in economics dealing with optimal taxation and it too has become a landmark. In a third totally different area, Ramsey was the originator of what has now come to be known as subjective probability theory, and these topics essentially exhaust his life-time output as he died of a stomach ailment at the age of 26.[2] It has been amusing for me, with a foot in both camps, to find out, via an informal poll, that almost all economists and mathematicians were completely unaware that Ramsey had done work in the other's area.

[2]Irrelevant but interesting: Ramsey's brother, Arthur Michael Ramsey, is still alive, age 74, and was Archbishop of Canterbury from 1961 to 1975.

Returning to the subject at hand, Chapter 6 deals with problems of Ramsey type for general technologies. The questions addressed are (i) existence of optimal trajectories, (ii) asymptotic behavior of weak trajectories (i.e. turnpike properties), (iii) when such trajectories admit characteristics and (iv) a result which relates optimal trajectories to the equilibrium theory of Chapter 5 by considering a model with a countable number of consumers, one for each time period.

As to an overall evaluation of the book, since I have already made some mildly unfriendly comments, I should hasten to state here that the handful of us who work in this area should certainly have a copy of the book. It contains things which are to be found nowhere else and, as mentioned, some of the original contributions of the authors are elegant and provocative. The outsider who wishes to get the flavor of this material would, I think, have a much easier time looking at the book of Nikaido [7] which covers a good deal of the same material. As a job of exposition I would give the book rather low marks, but in saying this I feel it is important to take account of the special circumstances. The co-authors are part of a group which was set up about ten years ago in Novosibirsk, headed up by Kantorovitch. Their research has heretofore appeared only in Russian language publications (only two of their twenty odd papers have been translated into English). This book serves as a means for the authors to let their co-workers in the West know what they have been up to over the past decade. As such it represents a useful step in attacking the difficult but important problem of achieving better east-west scientific collaboration in this area of mutual interest.

Finally, I must report a staggering number of typographical errors. I ran into them almost every time I tried to study a section closely to determine precisely what the authors had done. My assumption is that these are the fault of the translator rather than the authors. Also I doubt whether the Russian version said "In Chapter 5, we state and study a model of economic equilibrium" or "the set of trajectories of admissable trajectories that start from a given point could be numerous indeed." It's too bad to have this additional roadblock for the potential reader to have to overcome.

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Harmonic analysis on compact solvmanifolds, by Jonathan Brezin, Lecture Notes in Math., vol. 602, Springer-Verlag, Berlin, Heidelberg, New York, 1977, v + 177 pp.

Harmonic analysis on nilpotent and solvable groups is something of a stepchild in today's mathematical family. Lacking the charisma of semisimple harmonic analysis and the relative docility of abelian harmonic analysis, it receives patronage neither from the élite nor from the masses. Though it enjoys occasional largesse from a variety of donors it must rely for its main livelihood on the benevolent researches of a small number of faithful sympathizers. Even that most famous of nilpotent groups, the cornerstone of the subject, an object of which every modern mathematician should be aware, the Heisenberg group, is rather far from being a household word. And the extremely elegant core of the theory, the method of orbits, probably first adumbrated by Harish-Chandra, but really exposed by A. A. Kirillov, and further substantially developed by Auslander-Kostant and others, probably still counts as specialist's knowledge.[1] This relative obscurity must be considered more a vagary of history than a divine judgement on the subject's intrinsic merits, for nilpotent and solvable harmonic analysis offers problems of depth and classical precedent. The small hardy band mentioned above has been patiently working on some of these. In recent years much of their attention has been directed towards problems of analysis on compact solvmanifolds, that is, on compact quotient spaces $\Gamma \setminus S$ where S is a solvable Lie group and Γ is a discrete subgroup. (If S is nilpotent, then $\Gamma \setminus S$ is called a nilmanifold.) The book under review offers a substantial survey of the work on solvmanifolds done to date. One of the book's strong points, one which makes it a good entryway for those curious about the subject, is the large amount of space devoted to representative examples.

If f is a function (C^∞, L^p, you choose–that's part of the fun) on $\Gamma \setminus S$, then f may be regarded as a function on S invariant under left translation by elements of Γ. That is, $f(\gamma s) = f(s)$ for γ in Γ and s in S. There is a naturally defined action ρ of S on the functions on $\Gamma \setminus S$ by right translation: $\rho(s')f(s) = f(ss')$ for s, s' in S. The basic problem of harmonic analysis in this context is, given a reasonable space of functions, to find the subspaces which are invariant under $\rho(S)$ and are minimal with respect to this property

[1]Kirillov's basic result on nilpotent groups can be stated so succinctly that I can't pass up this opportunity to disseminate it. Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{N}$. Let Ad be the adjoint action of N on $\mathfrak{N}$. Let $\mathfrak{N}^*$ be the vector space dual to $\mathfrak{N}$ and Ad* the action of N contragredient to Ad. Then there is a canonical bijection between the set of equivalence classes of irreducible unitary representations of N and the collection of Ad*N orbits in $\mathfrak{N}^*$.