found and the objects detected but not yet verified. Optimal solutions which use this information cannot be determined with the methods used (see Chapter VI); policies which are "optimal" neglecting the feedback can be found, but simple adaptive policies are shown to improve on them.

(If there is a significant criticism of the work under review, it is that insufficient attention is paid to algorithms which achieve or approximate optima in relatively complex situations, as opposed to problems which admit of elegant solutions.)

I have surveyed essentially the first six chapters of the book but have not done justice to the thoroughness and clarity of the exposition nor to the numerous and helpful examples. The remaining chapters deal with approximations and with moving targets, for which some interesting results are found, though not of the same generality as the earlier work.

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Finite free resolutions, By D. G. Northcott, Cambridge Univ. Press, New York, xii + 271 pp., $29.50.

This book gives a beautifully self-contained treatment of the recent Buchsbaum-Eisenbud theory [4], [5] of finite free resolutions over a commutative ring with identity, as well as of a number of related topics (e.g. MacCrae's invariant [11]). There are two features in which the author's treatment differs from existing accounts of the subject: first, he confines himself almost entirely to elementary methods, avoiding Ext, Tor, and even exterior powers (we shall do likewise), and, second, he exploits a new notion of grade (or depth) in the non-Noetherian case which permits him to dispense entirely with the Noetherian restrictions on the ring. The very elementary form of the treatment enables the author to make accessible some fancy results from the homological theory of rings to readers with virtually no background in algebra.

Hilbert [7] gave the theory of finite free resolutions its initial impetus. Suppose that one is trying to understand a finitely generated module M over a Noetherian ring R (Noetherian means that every ideal is finitely generated, and implies that every submodule of a finitely generated module is finitely generated). To give generators $u_1, \ldots, u_{n_0}$ for M is essentially the same as to map a free module $F_0 = R^{n_0}$ onto M (the map then takes $(r_1, \ldots, r_{n_0})$ to $\Sigma_i r_i u_i$). To understand M, one simply needs to understand the kernel $\{(r_1, \ldots, r_{n_0}) \in R^{n_0}: \Sigma_i r_i u_i = 0\}$, call it $\operatorname{syz}^1 M$, of this map (of course it is not unique: it depends on the choice of generators). This kernel is called a *relation module* or *module of syzygies* for M. Note that $M \cong F_0/\operatorname{syz}^1 M$. But then, to understand $\operatorname{syz}^1 M$, it is entirely natural to choose, say, n_1 generators for $\operatorname{syz}^1 M$ (equivalently, to map $F_1 = R^{n_1}$ onto $\operatorname{syz}^1 M$) and so obtain a module of syzygies of the module of syzygies, denoted $\operatorname{syz}^2 M$. Of course, there is no reason to stop at this point, and so one can obtain a (usually infinite) sequence of modules of syzygies $\operatorname{syz}^i M$ each contained in a free module $F_{i-1} = R^{n_{i-1}}$. For each i we have a composite map $(F_i \twoheadrightarrow \operatorname{syz}^i M \hookrightarrow F_{i-1})$, call it d_i, and putting these together we obtain an exact sequence

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to M \to 0,$$

a so-called free resolution of M. If the F_i all have finite rank and $F_i = 0$ for all sufficiently large i, we call the resolution a *finite free resolution.*

Hilbert established the credibility of this notion by proving that if R is a polynomial ring over the complex numbers and M is a finitely generated graded module, then M does indeed have a finite free resolution: this is the famous Hilbert syzygy theorem. He even gave an algorithm for finding the resolution, and showed that its length (the subscript on the last nonzero F_i), if it is chosen of minimal length, does not exceed the number of variables. While he was at it, he proved that polynomial rings in finitely many variables are Noetherian. All this was motivated by the problem of showing that rings of invariants of matrix groups are finitely generated (later to become part of Hilbert's fourteenth problem). The restriction to the graded case is unnecessary: this has been known for a long time if one allows projective modules (i.e. direct summands of free modules) to play the role of the F_i. Recently, Quillen [13] and Suslin showed (independently) that all projective modules over polynomial rings over a field (or principal ideal domain) are free.

Over sixty years later the theory of finite free resolutions received a second enormous boost from the work of Auslander, Buchsbaum, and Serre [1], [2], [14]. To describe their results as simply as possible we shall want to assume that the ring R is local, i.e. has a unique maximal ideal. A suggestive example, which motivates the terminology, is the ring of germs of continuous (or C^∞, or analytic) real or complex valued functions at a point of a topological space (or differentiable manifold, or analytic space). The unique maximal ideal consists of germs of functions which vanish at the point. Any commutative ring R (we always assume the presence of a multiplicative identity) has lots of local rings associated with it, one for each prime ideal P, obtained by adjoining as universally as possible multiplicative inverses for the elements not in P. By and large, the theorems about finite free resolutions over an arbitrary ring R can be reduced to the corresponding statements about local rings of R by standard kinds of tricks, and the theorems tend to have a simpler statement in the local case. Moreover, when R is local, every projective module is free [10].

We can now describe the results of Auslander-Buchsbaum [1] concerning finite free resolutions over a Noetherian local ring R with maximal ideal m. It turns out that the existence of modules possessing finite free resolutions over such a ring R is intimately connected with the existence of so-called regular sequences. If $E \neq 0$ is a finitely generated R-module, $x_1, \ldots, x_n \in m$ is called a *regular sequence* on E or an *E-sequence* if x_1 is not a zerodivisor on E and for each i, $1 \leq i \leq n - 1$, x_{i+1} is not a zerodivisor on $E/(x_1, \ldots, x_i)E$. For example, if $R = K[[x_1, \ldots, x_n]]$, the formal power series ring in n variables over the field K, then $x_1, \ldots, x_n$ is an R-sequence. It is a theorem that all maximal E-sequences have the same length, and this length is referred to as the *grade* of E.

On the one hand, if $x_1, \ldots, x_n$ is an R-sequence, then $R/(x_1, \ldots, x_n)R$ has a finite free resolution. In fact, let F_i be the free module on $\binom{n}{i}$ generators

$U(j_0, \ldots, j_{i-1})$, $1 \leq j_0 < \cdots < j_{i-1} \leq n$, and define d_i by

$$d_i(U(j_0, \ldots, j_{i-1})) = \sum_{t=0}^{i-1} (-1)^t x_{j_t} U(j_0, \ldots, \hat{j_t}, \ldots, j_{i-1}),$$

where $\hat{}$ indicates omission. Then

$$0 \to F_n \xrightarrow{d_n} \cdots \to F_1 \xrightarrow{d_1} \cdots \to F_1 \xrightarrow{d_1} F_0 \to R/(x_1, \ldots, x_n)R \to 0$$

is a finite free resolution.

If a module E over a ring R has a projective resolution of finite length, the length of a shortest such resolution is denoted $\mathrm{pd}_R E$ (the *projective dimension* of E over R). In the above example it turns out that no shorter free (free and projective are equivalent in the local case) resolution exists, and so $\mathrm{pd}_R(R/(x_1, \ldots, x_n)R) = n$.

On the other hand, if a Noetherian local ring R possesses a finitely generated module $E \neq 0$ such that $\mathrm{pd}_R E = n$, then R must possess an R-sequence of length n. The fundamental (and much more precise) theorem along these lines is the result of Auslander-Buchsbaum [1] which asserts:

If R is a Noetherian local ring and $E \neq 0$ is a finitely generated R-module which has a finite free resolution, then

$$\text{grade } R = \mathrm{pd}_R E + \text{grade } E.$$

Since all terms are nonnegative, we have in particular that $\mathrm{pd}_R E \leq$ grade R.

We note the following example: If $R = K[[x_1, \ldots, x_n]]$ and $E = R/(x_1, \ldots, x_d)R$, then $\mathrm{pd}_R E = d$, grade $E = n - d$ ($x_{d+1}, \ldots, x_n$ is a maximal E-sequence) and grade $R = n$.

A Noetherian local ring is called *regular* if its maximal ideal is generated by a regular sequence (on the ring). Formal and convergent power series rings over a field are good examples. Geometrically, a point of an algebraic variety over an algebraically closed field or of an analytic variety is smooth ($\equiv$ simple $\equiv$ nonsingular) if and only if the local ring (of germs of appropriate functions) at the point is regular. Among the early triumphs of the thoery of finite free resolutions are the proof by Auslander-Buchsbaum-Serre [1], [14] that a Noetherian local ring is regular if and only if every finitely generated module has a finite free resolution, and the proof by Auslander-Buchsbaum [2] that every regular local ring has unique factorization: this had been a longstanding problem.

We now jump to the present. There are several remarkable new results, due to Buchsbaum and Eisenbud, which bring deeper insight into the connection between R-sequences and existence of finite free resolutions.

A special case of the first of their theorems gives a criterion (derived from the acyclicity lemma of Peskine-Szpiro [12]) for a finite free complex $0 \to R^{n_d} \to^{A_d} \cdots \to^{A_1} R^{n_0} \to 0$ to be acyclic (i.e. exact, e[illegible]t possibly at R^{n_0}). Here, we assume that the n_i are positive integers and that A_i denotes the n_i by n_{i-1} matrix of the ith map for $1 \leq i \leq d$. (The condition for this sequence to be a complex is that $A_i A_{i-1} = 0$, $1 \leq i \leq d$.)

Now, if R were a field one could give an acyclicity condition purely in terms of the (determinantal) ranks r_i of the matrices A_i: to wit, for each i,

$1 \leq i \leq d$, $n_i = r_{i+1} + r_i$ (where r_{d+1} is defined to be 0). For an arbitrary Noetherian local ring, which has a much more complicated ideal structure than a field, this condition fails miserably. However, there is a beautiful pair of necessary and sufficient conditions [4] as follows:

(i) for each i, $n_i = r_{i+1} + r_i$, *and*

(ii) for each i, $1 \leq i \leq d$, the ideal generated by the r_i size minors of A is either the unit ideal, or else contains an R-sequence of length at least i.

Here we see the intimate relationship between R-sequences and finite free resolutions revealed in a particularly concrete way.

Northcott's book gives a second result of Buchsbaum-Eisenbud [5] in detail which we shall only mention briefly here: the result gives a factorization theorem for certain matrices of minors derived from the matrices occurring in a finite free resolution. It turns out that unique factorization in regular local rings is an immediate corollary. There are expositions in [6] and [9].

When the assumption that R is Noetherian is dropped, great difficulties arise. A key point in the Noetherian case is that if I is an ideal of R, E is a finitely generated R-module, and every element of I is a zerodivisor on E, then there is an element u in E, not 0, and a prime ideal P containing I such that $\mathrm{Ann}_R u = P$. In particular, we can choose $u \neq 0$ so that $Iu = 0$. Counterexamples are easy to construct if the finiteness hypotheses are dropped (e.g. every element of m is a zerodivisor on $E = \bigoplus_{x \in m} R/xR$, but no single nonzero element $u \in E$ is killed by m, unless m is principal).

Now, the condition, for example, for $0 \to R \to^A R^n \to 0$ to be acyclic, where $A = [a_1 \ldots a_n]$, is that there not exist $u \neq 0$ in R with $a_i u = 0$, $1 \leq i \leq n$. In the Noetherian case this condition implies (in fact, is equivalent to) the condition that the ideal $(a_1, \ldots, a_n)R$ contains a nonzerodivisor, but this is false in the non-Noetherian case. A non-Noetherian theory of grade was introduced in [3] and studied in [8] to overcome this difficulty. The idea is to define the "true grade" as the supremum of lengths of regular sequences which occur after allowing adjunction of indeterminates. In the present example, for instance, if we enlarge R to $R[x]$, $a_1 + \cdots + a_n x^{n-1}$ is a nonzerodivisor.

Professor Northcott has systematically exploited this idea and shown that with this notion of grade, virtually all the theorems from the Noetherian case work in general.

The book is extremely well written and there is a large set of Exercises dispersed through the text which are completely solved at the ends of the various chapters.

This book should be of great value to anyone with little background in algebra who wishes to plunge directly into the homological theory of modules over commutative rings.

REFERENCES

1. M. Auslander and D. A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405.

2. ______, *Unique factorization in regular local rings*, Proc. Nat. Acad. Sci. **45** (1959), 733–734.

3. S. F. Barger, *Generic perfection and the theory of grade*, Thesis, University of Minnesota, 1970; also Proc. Amer. Math. Soc. **36** (1972), 365–368.

4. D. A. Buchsbaum and D. Eisenbud, *What makes a complex exact?*, J. Algebra **25** (1973), 259–268.

5. ______, *Some structure theorems for finite free resolutions*, Advances in Math. **12** (1974), 84–139.

6. J. A. Eagon and D. G. Northcott, *On the Buchsbaum-Eisenbud theory of finite free resolutions*, J. Reine Angew. Math. **262/263** (1973), 205–219.

7. D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), 473–534.

8. M. Hochster, *Grade-sensitive modules and perfect modules*, Proc. London Math. Soc. (3) **29** (1974), 55–76.

9. ______, *Topics in the homological theory of modules over commutative rings*, CBMS Regional Conf. Ser. in Math., no. 24, Amer. Math. Soc., Providence, R.I., 1975, pp. 1–75.

10. I. Kaplansky, *Projective modules*, Ann. of Math. (2) **68** (1958), 372–377.

11. R. E. MacCrae, *On an application of Fitting invariants*, J. Algebra **2** (1965), 153–169.

12. C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. Inst. Haute Étude Sci; Paris, no. 42, 1973, 323–395.

13. D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

14. J.-P. Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proc. Internat. Sympos. on Algebraic Number Theory, Tokyo, 1955, pp. 175–189.

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Introduction to ergodic theory, by Ya. G. Sinai, Princeton Univ. Press, Princeton, New Jersey, 1977, 144 pp., $6.00.

The author has endeavored to present the general results of ergodic theory by examining special cases. His very considerable success testifies to the care and insight with which his examples, illustrating the methods and basic concepts of ergodic theory, have been chosen. The examples are, moreover, explained very clearly and at a level which should make the book accessible to a wide audience. The reader should be warned, however, that some of the results appear on first reading to be simpler than they really are, and that not all areas of ergodic theory are treated. The last section of this review will discuss a particularly important omission.

Ergodic theory arose from efforts to abstract some mathematically interesting aspects of dynamical systems. Two such systems, which are very closely connected, may be studied as examples. Consider first an ideal gas whose molecules are subject to the laws of classical mechanics and which are enclosed in a container. Statistical mechanics consists of the study of this system, and especially of the limiting behavior of its properties as the number of molecules tends to infinity. As a second example, consider a planetary system also subject to the laws of classical mechanics. Celestial mechanics deals with the study of such planetary systems. The second example differs from the first merely in that the case of interest is not the limiting one, and in that there are no collisions against the walls of a container. Ergodic theory is, to a large extent, the study of ideas which have their origin in statistical or celestial mechanics.

We proceed now to the concept of phase space, which has come to be a crucial idea in the study of dynamical systems. Phase space does not correspond to the physical space of the dynamical system. It is rather a representa-