misleading: In decomposition theory random variables and vectors figure only in terms of their laws, and the theory, while its origin is probabilistic, is purely analytical. However, those defects – at least in the eyes of the reviewer –are of very little importance. For the book ought to be considered as a classic–the best of its kind. It is well written and very instructive.

The total impression about the state of the theory is somewhat disturbing. The ingenuity and power of the methods and the great wealth of results still leave the basic problem unsolved: Find applicable general criteria so that, given a law one can find all its components, and, in particular, find whether it is an indecomposable or an I_0-law. It is hoped that the Linnik-Ostrovskiĭ book will serve as a catalyst for further search in this direction.

The untimely death of Linnik was a great loss for mathematics and for those who knew him.

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Completeness and basis properties of sets of special functions, by J. R. Higgins, Cambridge Tracts in Mathematics, no. 72, Cambridge Univ. Press, Cambridge, London, New York, Melbourne, 1977, x + 134 pp., $19.95.

The questions considered in this book arise from our wanting to represent a given function as a linear combination of particularly interesting or useful auxiliary functions–for example, the eigenfunctions of a boundary value problem. In this setting the idea has been traced back to Daniel Bernoulli, who used the expansion as a formal device; it was Fourier who showed that (sometimes) the formal solution is really a solution. There are natural questions to ask about Fourier series (apart from their use in solving eigenvalue problems): Does the series converge? Does it converge to the function we got it from? If so, is it the only series of its kind that represents that function? A collection of functions φ_n such that every function f (in a suitable class) has a unique expansion $\Sigma a_n\varphi_n$ that converges (in a suitable topology) to f is called a basis. This notion, when formulated in abstract terms, can be considered in any Banach space, or even in more general spaces; a given set $\{\varphi_n\}$, regarded as abstract elements, may or may not form a base depending on which space they are taken to belong to. Thus for example the trigonometric functions $\{e^{inx}\}$ form a basis in L^2 (periodic functions of integrable square) but not in C (continuous functions under uniform convergence). The trigonometric functions also form an orthogonal set, but this is only a feature that is convenient for computing the coefficients in the expansion, not an essential part of the idea of a basis. Most of the familiar separable Banach spaces turn out to have bases, but we know (only since 1973) that there are separable Banach spaces that have no bases [5].

A similar idea entered mathematics in a different way and beginners sometimes confuse it with the idea of a basis. In abstract terms, a set $\{\varphi_n\}$ of elements of a Banach space is called total if every element of the space can be represented as the limit of a sequence of finite linear combinations of the φ_n, i.e. as $\lim_{N\to\infty}\Sigma_1^N a(k, N)\varphi_k$ rather than as $\lim_{N\to\infty}\Sigma_1^N a(k)\varphi_k$. This is the property expressed, for the trigonometric functions or for the powers $\{x^n\}$, by the Weierstrass approximation theorem. On the whole, at least for applications, totality seems to be a more significant notion that basicity (a term I have borrowed from chemistry). Something of the difference is suggested by the following story. In the early days of radar, people approximated functions by the partial sums of their Fourier series, but were bothered by persistent spikes on the graphs. The presence of these spikes had in fact been known to mathematicians for some 40 years under the name of the Gibbs phenomenon; it reflects the nonuniformity of the convergence, in other words, the nonbasicity in space C. When the radar people changed to suitable different linear combinations of the trigonometric functions, the difficulty disappeared; indeed, the trigonometric functions are total in C.

At this point it is useful to issue a warning that although hardly anyone calls a basis by any other name, the term "total" has many synomyms, most of which have also been used as names for different concepts. Some of the confusion has arisen because there is a closely related concept called completeness, which for a set $\{\varphi_n\}$ in Hilbert space says that the zero element is the only element orthogonal to all the φ_n; and in Hilbert space, completeness is equivalent to totality. ("Complete" is not a really good term, since it is completely different from other uses of "complete", for instance in "a complete metric space"; but I follow tradition–and the book.) Unfortunately this concept of completeness can be generalized to spaces in which it is *not* equivalent to totality. The subject has consequently suffered badly from what I like to think of (not quite accurately) as Humpty Dumpty's principles [4] and Ko-Ko's law [6]: if a concept has a name, give it a different one; if a concept hasn't a name, give it the name of something else; if two concepts have the same name, they can be identified.

The connection between completeness and totality was recognized long before Banach spaces had been invented, and rather naturally, because there are many reasons for wanting to know whether a given set is total (or complete): for example, to validate an algorithm for a physical problem (as for Fourier series); to answer a significant theoretical question (as for the Weierstrass approximation theorem); or to satisfy intellectual curiosity (when is a sequence $\{e^{-t}t^{\lambda_n}\}$ complete?). Furthermore, although the sets of functions that arise from physical problems often turn out to be bases or at least total sets, there is always (until some sufficiently general theorem has been established) the nagging possibility that next time it might turn out differently. Many special sets of functions have been investigated for basicity or totality and many methods are available for studying them. Confronted with an unfamiliar set of functions, where does one go for information? There has been no easy answer. There are two books [7], [8] specifically devoted to bases, but to bases (and their generalizations) in very abstract terms, so that neither book contains an appreciable number of examples of specific sets of functions that form bases. Moreover, since the theory of bases in abstract spaces has now taken on a life of its own, these books contain few theorems that an outsider can readily apply to a specific case. Information about particular bases and particular total sets can be extracted, but not very conveniently, from almost any book on approximation theory, from books on series of orthogonal functions, and even from books on functional analysis; but there has been no book devoted to this subject.

The book under review tries to fill this gap in the literature. It sets out to describe and illustrate some important methods for establishing the basicity or totality of a given set of functions. After a chapter reviewing metric spaces and L^p spaces, the author first discusses orthogonal systems and the special criteria that apply to them. He establishes the completeness of many named sets, and also of the (as yet unnamed) set $\{\pi^{-1}(x-n)^{-1}\sin \pi(x-n)\}$ that appears in the cardinal interpolation series (otherwise known as Shannon's sampling theorem), and he discusses some cases of completeness in the complex domain. He does not, however, mention the Franklin set, which recently turned out to be essential in Bočkarev's construction [3] of a basis for the Banach space of functions analytic in a disk and continuous in the closed disk.

The next chapter considers nonorthogonal systems. The author approaches them first through various versions of the stability principle, which states (to put it informally) that a sequence close to a basis is again a basis. One version of this principle was discovered, and exploited rather spectacularly, by Paley and Wiener, and has been extensively generalized; for a particularly neat presentation see [2]. The author then presents a very different approach via complex analysis and transform theory; this is an even older method which has been very successful in establishing completeness rather than basicity. A final chapter gives an introduction to boundary value problems that have complete orthogonal systems of eigenfunctions (these help to explain why so many familiar sets are bases or at least total sets). There is a useful three-page appendix tabulating 44 systems of functions and their completeness properties.

The book is admittedly an introduction which aims only to show the reader a few important systems and a few important methods. The author's choice of methods can hardly be faulted: the most widely used methods are indeed there. The applications are carefully worked out, sometimes in what seems to be excessive detail. (The author did miss Bourgin's work [1] on completeness of sets $\{f(nx)\}$, which uses methods rather different from any discussed in the book.) Since, like any introduction, the book is necessarily limited in scope, a reader who becomes interested in the field to the point of wanting to do research in it must not assume that this book describes the frontiers of the field, even for the quite concrete problems that it discusses.

The book does indeed help to fill the gap it was intended to fill. However I found it rather disappointing. Compared to classical Cambridge Tracts like those of Ingham, or Hardy and Rogosinski, or Smithies, it contains surprisingly little material, treated rather unevenly. The classical Tracts were written by mathematicians who knew everything about their subjects; they could pick out the most illuminating topics and present them economically and elegantly. This author just seems to be working too close to the limits of his own knowledge to be able to produce a really effective presentation or to make the best choice of material.

There is a regrettable tradition, whose rationale I cannot discover, that

British mathematical books have wretched indices. This one maintains the tradition.

References

1. R. C. Buck, *Expansion theorems for analytic functions*. I, Lectures on Functions of a Complex Variable (W. Kaplan, M. O. Reade and G. S. Young, Editors), Univ. of Michigan Press, Ann Arbor, 1955, pp. 409–419; p. 410.
2. D. G. Bourgin, *A class of sequences of functions*, Trans. Amer. Math. Soc. **60** (1946), 478–518.
3. S. V. Bočkarev, *Existence of a basis in the space of functions analytic in the disk, and some properties of Franklin's system*, Mat. Sb. **95** (137) (1974), 3–18, 159.
4. L. Carroll, *Through the looking-glass*, Chapter VI (In The Complete Works of Lewis Carroll, Nonesuch Press and Random House, London and New York, n.d.).
5. P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. **130** (1973), 309–317.
6. W. S. Gilbert, *The Mikado*, Act 2 (In The Savoy Operas, Macmillan, London, 1926, p. 371).
7. J. T. Marti, *Introduction to the theory of bases*, Springer-Verlag, New York, 1969.
8. I. Singer, *Bases in Banach spaces*, Springer, Berlin, 1970.

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Brownian motion, Hardy spaces and bounded mean oscillation, by K. E. Petersen, London Mathematical Society Lecture Note Series (2) vol. 28, Cambridge University Press, Cambridge, London, New York, Melbourne, 105 pp.

In recent years the techniques and theorems of Brownian motion have been used to prove theorems about harmonic and analytic functions. It is always pleasant when two branches of mathematics which ostensibly have little to do with one another can help each other out. There are two main links which allow Brownian motion (roughly representing the paths of an idealized random traveller) to be connected to the theory of harmonic and analytic functions. Kakutani [4] showed that Brownian motion can be used to solve the Dirichlet problem. Dispensing with the technicalities of continuity, smoothness, and measurability, here is what Kakutani's theorem says: Let S be an open set in $\mathbf{R}^n$ and let u be a real-valued function defined on ∂S. Let $z \in S$ and consider a typical Brownian path γ_z starting at z. Let $s(\gamma_z)$ denote the point of ∂S at which γ_z first hits ∂S. Define $\hat{u}(z)$ to be the average value of $u(s(\gamma_z))$, where the average is taken over all Brownian paths γ_z. Then $\hat{u}$ is a harmonic function on S with boundary values u.

A theorem of Lévy [5] links Brownian motion to analytic functions defined in the plane. This thoerem states that a nonconstant analytic function composed with Brownian motion is also Brownian motion, although the time scale must be changed on each Brownian path. The inituition behind Lévy's result is that an analytic function preserves angles, so that the randomness of direction is preserved. Since an analytic function need not preserve lengths, an adjustment of the time scale is necessary.

For $0 < p < \infty$ and u a function defined on the open unit disk D of the complex plane, define