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BOOK REVIEWS

Nonlinear operators and differential equations in Banach spaces, by Robert H. Martin, Jr., Wiley, New York, 1976, xi + 440 pp., $27.50.

Ordinary differential equations in Banach spaces, by Klaus Deimling, Lecture Notes in Math., vol. 596, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 136 pp.

In his celebrated lecture at the Paris Mathematical Congress of 1900, Hilbert asked whether the expansion of mathematical knowledge would not finally make it impossible for the single researcher to grasp all parts of it. As partial answer he said that all true progress goes hand in hand with the development of sharper aids and simpler methods, which make it easier to understand earlier theories and to circumvent complicated older procedures.

It would be hard to find better grounds for Hilbert's optimism than are given by current progress in the field of abstract differential equations. An excellent account of this current progress can be found in Martin's book (1976), also in *Ordinary differential equations in Banach spaces*, 1977, by Klaus Deimling. The latter is volume 596 of the Springer Lecture Notes, but is a much more scholarly and polished job than the category "lecture notes" would lead one to expect. Both Martin and Deimling are warmly recommended by this reviewer. They are complementary rather than competitive, and the pleasure of reading one is enhanced by reading the other.

A major difference in the two texts is that Martin includes background material which can be found in other books, while Deimling has chosen to emphasize current developments only. The former point of view has pedagogical advantages, but it reduces the scope below what one might hope for from the book's size and from the dust-jacket description. Thus, "the important, fundamental techniques of nonlinear functional analysis" could include the notion of topological degree, bifurcation theory, general nonlinear semigroups, and variational inequalities, all of which are, in fact, excluded. (This is said in no critical spirit, but simply as information.) Aside from a discussion of the connection between spectrum and numerical range in Chapter 3, and a development of fixed-point theory with application to Hammerstein equations in Chapter 4, the main strength of the book for research mathematicians would seem to be in the field of abstract differential equations. This is a field to which Martin (and also Deimling) have made fundamental contributions.

Let us begin with the differential equation $u' = f(t, u)$ where u is a function from an interval to a real Banach space X. When f is Lipschitzian, existence and uniqueness follow much as in the classical case $\dim X < \infty$, in which case the equation is a finite system. By standard techniques (Dugundji, Lasata-Yorke) the above remark gives approximate solutions when f is only continuous. However one cannot expect an exact solution under that hypothesis, because every proof of Peano's existence theorem makes essential use of compactness. In fact, counterexamples are well known (Dieudonné, Yorke, Godunov).

Additional hypotheses ensuring existence generally involve a function ω: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that the scalar equation $\rho' = \omega(t, \rho)$ has suitable properties; these properties need not be spelled out here. The simplest conditions are of the form

$$|f(t, x) - f(t, y)| \leqslant \omega(t, |x - y|). \tag{1}$$

However, it has been known at least since McShane's work (1939) that a one-sided condition often suffices. This remark in the abstract case leads to the rich theory of dissipative, accretive or monotone operators. (Terminology is not uniform and, in particular, "monotone" does not mean "monotone in the sense of Collatz.") The appropriate condition is now of the form

$$(p - q, x - y) \leqslant |x - y|\omega(t, |x - y|) \tag{2}$$

where $p = f(t, x)$, $q = f(t, y)$ and $(a, b) = (a, b)_-$ or $(a, b)_+$ is the semi-inner-product; cf. Lumer (1961). (The semi-inner-products are defined by means of the duality mapping, or also by means of right- or left-handed derivatives of the norm. If the norm is generated by an inner product, both semi-inner-products coincide with it.) Once a local solution is obtained one can declare that $u < v$ if v is an extension of u; then Zorn's lemma gives a global solution on a maximum interval of existence.

In the theory of evolution equations and nonlinear semigroups it is almost essential to let f be multivalued, so that the equation becomes $u' \in f(t, u)$; in fact, an interesting book by H. Brézis is formulated entirely in that language. The inequality (2) is now required for $p \in f(t, x)$, $q \in f(t, y)$. Central results here or in the single-valued case are due to Crandall, Brézis, F. Browder, Pazy, Benilan, Lakshmikantham, Phillips, Yorke, Kato, Liggett, Martin, Murakami, Pulvirenti, Knight, Fitzgibbon, Chow and Schuur, and Eisenfeld.

If the trajectory $u(t)$ is required to stay in a closed set $\Omega \subset X$, we denote the distance from x to Ω by $|x, \Omega|$ and introduce the following condition of Nagumo (1942):

$$\liminf_{h \to 0+} h^{-1}|x + hf(t, x), \Omega| = 0 \qquad (x \in \Omega). \tag{3}$$

To get by with a hypothesis in Ω, not involving $X - \Omega$, one projects the corners of the Euler broken-line approximation u_ε onto Ω at each step. It is then necessary to show that the interval of existence of u_ε does not tend to 0 with ε; this involves a subtle proof that (3) holds uniformly under natural hypotheses. Although important concrete cases were considered by Nagumo, Crandall and Hartman, the principal results in the abstract case are due to Martin; they are, on the whole, definitive. As an added bonus, note that existence of a solution in Ω plus uniqueness for the initial-value problem implies invariance of Ω.

These results apply to problems that, apparently, have little to do with differential equations; for example, to the proof that $T(X)$ is dense in X for certain classes of accretive operators T and to the existence of fixed points (Browder, Deimling, Martin, Crandall, Vidossich). A clever idea of P. Volkmann gives Browder's theorem to the effect that, if Ω is a locally closed subset of X, the set of exposed points of Ω is dense in $\partial\Omega$; this implies a well-known theorem of Bishop and Phelps on convex sets. The gist of Volkmann's argument is to construct a solution which escapes from Ω, and to show that it could not escape, by (3), if the desired conclusion were false.

To generalize the theorem of Peano, one can take $f = g + h$ where g is completely continuous and h Lipschitzian. A different generalization involves hypotheses of the form

$$\alpha f(t, \Omega) \leq \omega(t, \alpha\Omega),$$

where $\Omega \subset X$ and where α is the Kuratowski measure of noncompactness. Interesting results here have been obtained by Cellina, Szufla, Ambrosetti, Corduneanu, Pianigiani, Danes, De Blasi, Eisenfeld, Lakshmikantham, Bernfeld, and Deimling; the subject is not yet closed.

When $u' = f(t, u)$ is written in integral form it leads to an equation of Hammerstein type, $x + KFx = 0$. In the classical case F is a substitution operator and K a linear integral operator; these are associated with the names of Nemytskii and Urysohn, respectively. However, in a far-reaching generalization, Minty brings the subject into contact with variational inequalities

$$(y - x, G(x)) \geq 0, \qquad y \in \Omega \subset X,$$

and obtains existence by use of the Knaster-Kuratowski-Mazurkiewicz theorem. This in turn follows from Sperner's lemma, just as the Brouwer theorem and its Schauder-Leray generalization also follow. In the last analysis, then, it could be said that these major results of abstract mathematics are obtained by a sophisticated process of counting! Minty's approach gives new insight into the meaning of "coercivity" and generalizes a large literature.

The importance of variational inequalities is also underlined by the splendid work of Kinderlehrer on solutions of partial differential equations in the presence of obstacles. A somewhat different class of variational problems have the peculiarity that their solutions are functions of compact support. Such problems have been stuided by Berkowitz and Pollard, Brézis, Browning, Friedman, and, using his theory of quadratic forms in Hilbert space, by Hestenes. A useful tool in this study is an extension of the Hadamard-Littlewood derivates theorem to functions $\mathbf{R} \to X$ which greatly sharpens classical results.

So far, we have stressed existence. Let us now outline a few results in error estimation and stability. If $\delta(t) = |u(t), \Omega|$ and (2) and (3) hold, then, in general,

$$|u' - f(t, u)| \leq \varepsilon \Rightarrow \delta' \leq \omega(t, \delta) + \varepsilon, \tag{4}$$

where δ' denotes a suitable Dini derivate. If there exists a point $y \in \Omega$ nearest to x, the "tangent condition" (3) can be replaced by the "normal condition"

$$(x - y, f(t, y))_+ \leq 0. \tag{5}$$

Condition (5) was introduced by Bony in his study of the sharp maximum principle for partial differential equations. An inequality similar to (4) holds for solutions of broad classes of nonlinear parabolic systems, not indeed universally, but at the point x where $\delta(t, x) = |u(t, x), \Omega|$ attains its maximum. This observation, due in the main to W. Walter, generalizes invariance theorems of Weinberger, Bebernes and Schmitt, Chabrowski, and Haar. Further generalization, by Lemmert and Volkmann, allows the functions to have values in X, and also weakens the assumptions on the second derivatives; this weakening is of interest even in the classical theory of partial differential inequalities.

A curious feature of the multivalued case $u' \in f(t, u) + e(t)$ is that *one* element of $f(t, u)$ can be used for getting u' and *another* can be used in (3) or (5). Similar results hold when $\delta(t) = |u(t) - v(t)|$ for approximate solutions u, v, and also when $|x|$ is replaced by a more general measure of magnitude $\|x\|$ or $V(t, x)$; this measure need not be real-valued. Corresponding to the use of Dini derivates when $n = 1$, one can let u' (as well as f) be multivalued; then $u'(t)$ denotes the set of all right-hand derived values at t and the equation is written in the form

$$u'(t) \cap f(t, u(t)) \neq \emptyset. \tag{6}$$

It is useful to distinguish conditions that hold "mod E" (except in an enumerable set) from those that hold "mod N" (except in a null set). For example, in this language the mean-value theorem reads as follows: Let $K \subset X$ be closed and convex. For $t \in [a, b]$, let u be continuous, let $u'(t)$ be nonempty mod E, and let $u'(t) \cap K$ be nonempty mod N. Then $(u(b) - u(a))/(b - a) \in K$. Among contributors to the general theory of invariance, estimation, and stability as sketched above are Nagumo, Mazur, Saks, Tapia, Collatz, Dieudonné, McLeod, Walter, Bony, Crandall, Lakshmikantham, Schröder, Murakami, and Volkmann. The latter's main result is one of the deeper theorems in this general area; it is stated without proof by Deimling and omitted by Martin.

Although there are precursors in the work of Walter and L. Elsner, the definitive formulation of quasimonotonicity in the abstract case is due to Volkmann. If Ω is an order cone in the sense of Krein, Volkmann's results can be interpreted as saying that Ω is invariant; hence, they are contained in (4). Here the continuity condition (1) or (2) can be replaced by

$$\|f(t,y) - f(t,x)\| < \omega(t, |y - x|) \quad \text{when } y > x,$$

where $\|\cdot\|$ is one of the Kamke norms which generate the order relation and where the cone is assumed to have nonempty interior. The first use of this surprisingly weak hypothesis is due to Walter, as is an interesting theorem asserting equivalence of Volkmann's quasimonotonicity condition with the Nagumo tangent condition for the order cone. (Walter's proof has been simplified by Deimling.) Also due to Walter is a striking use of quasimonotone systems in the study of nonlinear parabolic equations and boundary-layer theory; this "line method" has recently been extended to unbounded regions by Voigt. The theory sketched above generalizes a substantial literature, going back to M. Müller (1926).

An important application of these techniques is to relations

$$z'(t) \cap f(t, z(t)) \neq \varnothing,$$

where z is a linear operator on a complex Hilbert space. If Ω is the Siegel disk $z^*z \leqslant 1$, the tangent condition is often needed only on the Šilov boundary; this remark greatly increases the scope of the results. The special case

$$a(t) + b(t)z(t) + z(t)d(t) + z(t)c(t)z(t) \in z'(t)$$

applies to equations of multiple transmission lines and transport processes, and also yields results on pure operator equations (no derivatives). For example, if $b \neq 0$ and $d \neq 0$, then one of the functions

$$f(z) = a + bz(1 - cz)^{-1}d, \quad g(z) = c + dz(1 - az)^{-1}b$$

maps the Siegel disk into itself if, and only if, the other one does. Further study of operator differential equations gives results on oscillatory properties of $(pz')' + qz = 0$ which parallel those in the classical case. Extension to higher-order equations involves a far-reaching generalization of the notion of "adjoint" where, instead of an adjoint operator, one has an adjoint subspace. Among contributors to these developments are Ambartzumian, Preisendorfer, Reid, Bellman, Kalaba, Wing, Ueno, Chandrasekhar, A. Wang, Zakhar-Itkin, J. Levin, Paszkowski, Shumitzky, Helton, Krein and Shmul'yan, Etgen and Lewis, and Coddington and Dijksma.

R. M. REDHEFFER

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Nonlinear semigroups and differential equations in Banach spaces, by Viorel Barbu, Noordhoff International Publishing, Leyden, The Netherlands, 1976, 352 pp.

The typical first graduate course in ordinary differential equations begins with a discussion of the initial-value or Cauchy problem. Under a variety of assumptions, it is shown that this problem has a solution, that it is unique, and that it depends nicely on the data. Thus, under mild restrictions, Cauchy problems in classical ordinary differential equations are well posed. As the course progresses and more special topics are pursued, these preliminary results begin to seem rather simple and, in a short time, are taken for granted by the serious student. Nevertheless, one is always thinking in terms of them. Scientists and engineers often think the same way: a system being modeled has a state u which changes in time according to a differential (or evolution) equation

$$\text{(EE)} \qquad du/dt = A(u)$$

which summarizes the dynamics of the system. In classical mechanics and many other fields the state is a list of numbers (giving, e.g., velocities and positions of bodies or populations of species or quantities of reactants, etc.,) and (EE) is a classical ordinary differential equation, where "classical ordinary differential equation" means roughly that A continuously maps an open subset of some $\mathbf{R}^N$ into $\mathbf{R}^N$. One specifies an initial condition

$$\text{(IC)} \qquad u(0) = u_0$$