point not on X lying on a block with each pair of the three points. (Picture a tetrahedron.) Then each point y not in X produces a partition of X via the set of associated "tetrahedra", and by varying y a parallelism of X results (with $t = 3$). There are only two known examples of such biplanes $B(k)$, with $k = 3$ or 6. A beautiful result of the author is presented: if, in addition, every four points of X lie in a subbiplane $B(6)$, then the biplane can only be $B(6)$. This is proved by a brief argument, in which it is shown that the parallelogram property must hold, and hence that the parallelism is of known type. Additional questions involving parallelisms and biplanes lead to some very special association schemes and metrically regular graphs.

Lurking in the background throughout many of these topics are the group theoretic situations in which many of the combinatorial questions were originally asked. This connection is described in the next to the last chapter. Consider a parallelism of t-sets of X, and let G be its automorphism group. If G is $(t + 1)$-transitive on X and $|X| > 2t > 2$, then the parallelism is shown to have the parallelogram property (and hence is known) or to be of a unique type with $|X| = 6$, $t = 2$. This result, and its proof, are very typical examples of how large groups can be used in combinatorial situations: use of various stabilizers of one or more points of X leads to local "configuration" properties, which in turn permit purely combinatorial classification theorems to be applied. All the group-theoretic background required is proved (in yet another appendix); moreover, the group-theoretic question which required the preceding theorem is also presented. The discussion of automorphism groups concludes with a brief sketch of the classification of parallelisms for which G acts 2-transitively on the set of parallel classes.

Many open problems are presented throughout the book; indeed, the impression is clearly conveyed that any theorem, however beautiful and complete, easily leads to many problems. The final chapter discusses generalizations of the concept of parallelism. Naturally, large numbers of additional open problems result.

This book is a delight to read. The proofs are slick, but well motivated. It is short and carefully organized. The required background is minimal, being only part of a standard first year graduate algebra course. It would be an excellent way for a graduate student to learn many different techniques, some of which may be difficult, but all of which have clear cut and immediate applications to the main topic being studied.

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Lectures in semigroups, by M. Petrich, Akademie-Verlag, Berlin, 1977, viii + 168 pp.

What should a book on semigroups be about? Semigroups, of course, but the subject has long outgrown such a simple answer. In 1961 it was possible for Clifford and Preston [3] to attempt to be comprehensive, provided they stuck very strictly to the algebraic theory and ignored ordered and topological semigroups altogether; but by the time their second volume appeared in 1967 any hope of comprehensive cover in a book of reasonable size was gone for good. So in 1977 any aspiring author of a book on semigroup theory must select.

On what principles ought he to select? The question interests me because I have myself recently written a book on semigroups [7] whose contents are nearly disjoint from those of Petrich's book. One possible approach is to subordinate the semigroups to their applications and to concentrate on those aspects of the theory that have found applications–especially to automata, languages and machines. This approach can be seen in a number of books not primarily about semigroups, notably those by Arbib [1] and Eilenberg [4], but so far as I am aware no author has yet written a book solely on semigroups using the "applications" principle of selection, and this is certainly not Petrich's approach.

At the other extreme, one could select material for inclusion solely on the grounds that one found it interesting. To some extent all authors use this principle, including those seemingly motivated by applications–who presumably are so motivated because they find the applications interesting. But on its own the principle is a highly subjective one and most of us are happier if we can justify our selections in some other, more objective way.

Cohesiveness, I would suggest, is the key; and Petrich's new book, like his earlier book on semigroups [8] demonstrates the quality admirably. Semigroup theory, like other branches of abstract algebra, has inched forward in a highly disjointed and disconnected way. The very substantial task facing an author is to select and organize material so that a collection of papers written by authors with widely differing emphasis and often largely ignorant of each other's work becomes a coherent whole. This Petrich has managed very well.

The book is emphatically not for beginners, and anyone attempting to pick up the rudiments of the subject by reading the introductory Chapter I would find the going heavy. This is not to say that Chapter I is badly written, but only that its purpose is to introduce notations and conventions rather than to serve as a primer for the uninitiated.

The brief introductory chapter over, Petrich launches into an account of bands (semigroups satisfying $x^2 = x$) more comprehensive and complete than anything previously seen in print. Semigroups differ dramatically from rings in that the identical relation $x^2 = x$ does not imply commutativity, and it is only in relatively recent years that anything approaching a full understanding of noncommutative bands has been achieved. Petrich stops short of the Birjukov-Fennemore-Gerhard classification (see [2], [5], [6]) of varieties of bands, which may be a pity, since a reconsidered, carefully written account of this work, embodying the best features of the three independent accounts available, would have been a great service to the semigroup community. But if a volume is to remain reasonably slim a line has to be drawn somewhere, and in the space available Petrich gives an excellent account of bands, including not only a good deal on varieties but also some work on classes defined by implications.

If ϕ is a homomorphism from a semigroup S onto a band B, then each $b\phi^{-1}$ ($b \in B$) is a subsemigroup of S and so one can think of ϕ as providing a decomposition of S into disjoint subsemigroups. Advantage is gained if B is a band of a special sort, such as a semilattice. Much work has been done on semilattice decompositions, the most thorough account being in Petrich's earlier book [8]. Chapter III of the present book concentrates on *rectangular band* decompositions, for which the band B satisfies the identical relation $xyx = x$. These are called by Petrich *matrix* decompositions, reasonably enough, since the decompositions of S in this case is into disjoint subsemigroups S_{ij} indexed by a set $I \times J$ and such that $S_{ij}S_{kl} \subset S_{il}$.

Rectangular bands are in a natural sense opposite to, or rather complementary to semilattices, both because they are characterized by the "anticommutative" property $xy = yx \Rightarrow x = y$ and because if E is the maximum semilattice homomorphic image of a band B then the homomorphism ϕ: $B \twoheadrightarrow E$ decomposes B into rectangular bands. Thus, after semilattice decompositions, matrix decompositions are the most natural band decompositions to consider.

Within the variety of bands the join of the variety of semilattices ($xy = yx$) and the variety of rectangular bands ($xyx = x$) is the variety of *normal* bands ($xyzx = xzyx$). Thus normal band decompositions are the natural next step after semilattice and matrix decompositions. They form the subject of Chapter IV.

Chapter V is less closely linked in content and method with the chapters that go before, but its theme "Lattices of subsemigroups" proves unexpectedly rich in material. Here as elsewhere in the book Petrich reveals his encyclopaedic knowledge of semigroups, especially of the very extensive East European literature on the subject. Much of the material here is available to nonreaders of Russian for the first time.

In these days of cost-cutting it is noteworthy that the book is beautifully printed (in East Germany) rather than photocopied from a typescript. It is well written, in a fairly clipped, formal style, but clearly. Lovers of the English language who wince with me at the opening sentence of §I.1

> Of all the numerous generalizations of group or ring theory, the theory of semigroups has been undoubtfully the gratest success

should take courage and read on, for the unfortunate lapses in that sentence are quite uncharacteristic.

While the book is unlikely to have a wide readership, it will be very useful indeed to the semigroup specialists at whom it is aimed.

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Elliptic functions and transcendence, by David Masser, Lecture Notes in Math., vol. 437, Springer-Verlag, Berlin, Heidelberg, New York, xii + 143 pp., $7.80.

I. A history. The original proof of the transcendental nature of the number e by Hermite in 1873 was based on a delicate scheme of rational approximations which seemed to be applicable only to the exponential function. In this light one may view with a sympathetic eye Hermite's pessimism toward the problem of the transcendental nature of π, as he openly states in a letter to Borchardt (Crelle, vol. 76, p. 342): "Que d'autres tentent l'entreprise, nul ne sera plus heureux que moi de leur succès, mais croyez-m'en, mon cher ami, il ne laissera pas que de leur en coûter quelques efforts." A few years later in 1882 Hermite would be amazed by the remarkable simplicity of Lindemann's proof of the transcendentality of π based on Euler's identity $e^{\pi i} = -1$ and on Hermite's earlier ideas. This episode marks the exalting birth of the theory of transcendental numbers and was to represent the only significant contribution for some time. What followed in the next quarter of a century was no more than a generalization of ideas and a simplification of methods, first in the hand of Weierstrass who saw that the method of Hermite and Lindemann could be made to yield a proof of the algebraic independence of the values of the exponential function at distinct algebraic points; this was followed by technical simplifications by Gordan, Hilbert and Hurwitz.

By the end of the nineteenth century it was generally believed that the main arithmetical properties of the exponential function were well understood; there were good reasons for this. For one, the work of Kummer on cyclotomic extensions had been around for more than half a century, even though his methods were beginning to be forgotten; the work of Kronnecker on complex multiplication was being brought to completion. One knew well that the values taken by the exponential function $e^{2\pi iz}$ at the rational points on the projective line $\mathbf{P}^1(Q)$ were values at *special points*, i.e. they generate abelian extensions of the rationals and all such extensions arose in this way. One may surmise that in 1900 Hilbert, being thoroughly familiar with these properties of the exponential function after the manner of his Bericht, would have present in the back of his mind these results when formulating his Seventh Problem on the arithmetical nature of numbers of the form α^β and in particular of $2^{\sqrt{2}}$, and in his Twelfth Problem concerning the search for automorphic forms whose values at *special points* of certain moduli varieties would generate algebraic extensions of number fields with special Galois properties.