BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, March 1978

ON THE LATTICE OF FACES OF THE n-CUBE[1]

BY N. METROPOLIS AND GIAN-CARLO ROTA

Communicated by J. R. Goldman, August 8, 1977

1. We give a simple axiom system for the lattice L_n of faces of the n-cube, which is independent of dimension, and we construct a partition of the lattice into a minimum number of chains, or *Dilworth partition.* This partition turns out to enjoy some notable symmetries.

We use the representation of the faces of an n-cube as *signed subsets* of an n-set, say of the set $\{1, 2, \ldots, n\}$. A signed subset $A_\sigma = (A_1, A_2)$ is an ordered pair of disjoint subsets, where A_1 is called the *positive part,* and A_2 the *negative-part.* If B_σ is also a signed set, write $A_\sigma \leqslant B_\sigma$ when $A_1 \supseteq B_1$ and $A_2 \supseteq B_2$. Add a minimum element 0—the *improper face*—to the ordered set of signed sets, thereby making it a lattice L_n. The maximum element I of L_n is the signed set $I = (\phi, \phi)$. We use the terms "face" and "signed set" interchangeably.

On the lattice of signed subsets one defines *diagonals* $\Delta(A_\sigma, \cdot\,)$. For a given face A_σ, such a diagonal is a function defined on the segment $[0, A_\sigma]$ of L_n, and $\Delta(A_\sigma, B_\sigma) = C_\sigma$, where $C_1 = (A_1, B_2)$ and $C_2 = (A_2, B_1)$. On the improper face one sets $\Delta(A_\sigma, 0) = 0$. Geometrically, the diagonal $\Delta(A_\sigma, \cdot\,)$ associates to each face contained in A_σ the unique opposite face inside the face A_σ. When $A_\sigma = I$, the diagonal $\Delta(I, \cdot\,)$, written $\Delta(\,\cdot\,)$, is a cubical analog of complementation in a Boolean algebra.

2. **Main Theorem.** Let L be a finite lattice with minimum 0 and maximum I. For every $x \neq 0$, let Δ_x be a function defined on the segment $[0, x]$ and taking values in $[0, x]$. Assume: (1) if $y \leqslant x$, then $\Delta_x(\Delta_x(y)) = y$; (2) if $a \leqslant b \leqslant x$, then $\Delta_x(a) \leqslant \Delta_x(b)$; (3) if $a < x$, then $a \wedge \Delta_x(a) = 0$; (4) let $a < x$ and $b < x$. Then the following two conditions are equivalent: $\Delta_x(a) \wedge b < x$ and $a \wedge b = 0$. Then L is isomorphic to the lattice of faces of an n-cube for some n, and conversely.

3. **A symmetric Dilworth partition.** By Dilworth's theorem there exists a partition of L_n into $[n/3]$ chains. We explicitly describe one such partition, one that is invariant under the main diagonal Δ.

Consider a signed subset as a sequence $u_1 u_2 \cdots u_n$ whose *digits* u_i range over the alphabet $\{x, 0, 1\}$. Set $u_i = x$ if the element i is unsigned; otherwise,

AMS (MOS) subject classifications (1970). Primary 05B40; Secondary 05A17.

[1] Work supported by US Energy Research and Development Administration under Contract W-7405-ENG. 36 and partially supported by NSF MCS 7701947.

Copyright © 1978, American Mathematical Society

$u_i = 0$ or 1 according to whether u_i belongs to the positive or to the negative parts. We call 0 and 1 *significant digits* in contrast to x. A *vertex* is a sequence $u_1 u_2 \cdots u_n$ where no $u_i = x$.

The chains issuing from the vertices are constructed one step at a time, upwards from the vertices, according to the following rule:

(1) Change the last two digits of the vertex $u_1 u_2 \cdots u_n$, according to the following table: $11 \longrightarrow 1x$, $00 \longrightarrow 0x$, $10 \longrightarrow x0$, $01 \longrightarrow x1$. Denote the resulting face by $v_1 v_2 \cdots v_n$.

(2) Change the two rightmost adjacent significant digits of $v_1 v_2 \cdots v_n$ by the same rule.

(3) Repeat until the resulting face contains no two adjacent significant digits.

(4) Now the preceding algorithm assigns to chains issuing from vertices all faces $u_1 u_2 \cdots u_n$ for $u_1 = u_2 = x$, where u_j is a significant digit for some $j > 2$, and no two successive digits are significant.

(5) For faces containing three successive digits $x00$, $x01$, $x10$ or $x11$ use the following *bracketing algorithm*: (a) Bracket $x01$ to $(x01)$, etc., by inserting brackets at the appropriate places. In the resulting *bracketed face,* two digits are *successive* if they are successive in the sequence obtained after all bracketed digits have been removed. (b) Repeat the bracketing algorithm (a), for successive digits, until no unbracketed digits can be bracketed, and obtain a *completely bracketed* face. (c) On the set of completely bracketed faces having the same bracketed digits and the same bracket structure, remove bracketed digits and proceed as in steps (1)–(4), and then reinsert the bracketed digits in their original places. One obtains in this way a partition into chains of such a set. (d) The only remaining faces to assign to chains are those which have only x's in their unbracketed sequence, which is nonempty.

Let $u = u_1 u_2 \cdots u_n$ be one of these remaining faces. If $u = I$, the maximum element, assign it arbitrarily to any chain, say, the chain containing $0xx \cdots x$. If $u \neq I$, let u_k be the first significant digit, and let u_m be the first unbracketed digit, which must be an x. Then assign u to the chain containing $v_1 v_2 \cdots v_n$ where $v_m = u_k$ and $v_i = u_i$ otherwise.

The preceding technique for constructing Dilworth partitions can be extended to families of k disjoint subsets, ordered similarly to the case $k = 2$ of the cube, and even to certain sublattices of the lattice of partitions of a set. These constructions will be presented elsewhere.

BIBLIOGRAPHY

Harper and Rota, *Matching theory, an introduction,* Advances in Probability, Vol. 1 (Peter Ney, editor), Dekker, New York, 1970, pp. 171–217.

C. Greene and D. Kleitman, *Proof techniques in the theory of finite sets,* MAA Survey in Combinatorics (to appear).

UNIVERSITY OF CALIFORNIA, LOS ALAMOS SCIENTIFIC LABORATORY, LOS ALAMOS, NEW MEXICO 87545

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139