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HEREDITARILY PYTHAGOREAN FIELDS, INFINITE HARRISON-PRIMES AND SUMS OF 2^nth POWERS

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In [3] W. D. Geyer studied (infinite) algebraic number fields having an absolute Galoisgroup which is solvable as an abstract group. In particular he showed that for a real number field K of this type the absolute Galoisgroup $G(\bar{K}|K(i))$ must be abelian (we denote the algebraic closure of a field k by $\bar{k}$). Geyer's work may therefore be considered as a generalization of the well-known characterization of real-closed fields given by E. Artin and O. Schreier. This note reports on the work [1] originated in an attempt to carry over Geyer's results to arbitrary formally real fields K (= real fields). We investigate those fields with abelian Galoisgroup $G(\bar{K}|K(i))$ which may be regarded as substitutes for real-closed fields. The orderings of real-closed fields are to be replaced by certain infinite Harrison-primes, and the study of sums of squares by orderings can be extended with help of these Harrison-primes to sums of 2^nth powers.

1. Hereditarily pythagorean fields. A real field K is called *pythagorean* if $K^2 + K^2 = K^2$ holds, *hereditarily pythagorean* (= h. p.) if any real algebraic extension is pythagorean. Let $\mathbf{Z}_p$ be the compact additive group of the p-adic integers, δ_{ij} the Kronecker-symbol and $\mathrm{Br}(K)$ the Brauergroup of the real field K.

THEOREM 1. *K is a h.p. field iff $G(\bar{K}|K(i))$ is abelian. If K is h.p., then*

(i) $G(\bar{K}|K) = \langle\sigma\rangle \times G(\bar{K}|K(i))$, $\sigma^2 = 1$, *σ operates by inversion on* $G(\bar{K}|K(i))$,

(ii) $G(\bar{K}|K(i)) \cong \Pi_p\, \mathbf{Z}_p^{\alpha_p}$ *with* $\alpha_p = -\delta_{2p} + \dim_{\mathbf{F}_p} K^\times/K^{\times p}$,

(iii) $\mathrm{Br}(K)$ *has exponent* 2, $\dim_{\mathbf{F}_2} \mathrm{Br}(K) = \alpha_2 + \binom{\alpha_2}{2} + 1$

H.p. fields can further be characterized by the Haar-measure of the set of involutions in $G(\bar{K}|K)$ [1], by the existence of a certain henselian valuation [2] (both due to L. Bröcker), by the existence of a Kummer-theory for all algebraic extensions [1] (F. Halter-Koch) or by torsion properties of the Wittring of $K(X)$ [1].

2. Infinite Harrison-primes. An infinite Harrison-prime P [4] of K is called an *ordering of type $n \in \mathbf{N}$* if $K^{2^n} \subset P$ and of *exact type n* if $K^{2^n} \subset P$, $K^{2^{n-1}} \not\subset P$. Orderings of type 1 are the usual orderings. Let Q_n be the subset of all sums of 2^nth powers in K. Then $Q_n = \bigcap P$ where P ranges over all orderings of type n, the case $n = 1$ is due to E. Artin.

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Let $L|K$ be a field extension, P, $\widetilde{P}$ orderings of higher type of K, L respectively. $(L, \widetilde{P})$ extends (K, P) if $\widetilde{P} \cap K = P$ and P and $\widetilde{P}$ have the same exact type. In [1] an extension theory is established, for example: (i) the number r of extensions to L is less than or equal to $[L: K]$, (ii) $r = 0$ or $r = [L: K]$ if $L|K$ is a Galois-extension, in the latter case all extensions are conjugate. The extension-theory applies to the real closures $(R, \widetilde{P})$ (= the maximal algebraic extensions) of (K, P). Let P be of exact type $n \geqslant 2$, the case $n = 1$ is due to E. Artin and O. Schreier.

THEOREM 2. (i) *A real closure* $(R, \widetilde{P})$ *is a h.p. field with* $G(\bar{R}|R(i)) \cong \mathbf{Z}_2$, *has a henselian valuation with real-closed residue-class-field, two orderings of type* 1 *and a single ordering of exact type m for* $m \geqslant 2$,

(ii) *two real closures* (R_i, P_i), $i = 1, 2$, *of* (K, P) *are isomorphic iff* $R_1^{2^m} \cap K = R_2^{2^m} \cap K$ *for all* $m \in \mathbf{N}$.

Different from the usual Artin-Schreier-Theory there are in general infinitely many nonisomorphic real closures of a given (K, P). The proofs are essentially carried out by valuation theory since for an ordering P of higher type the set $\mathfrak{o}(P) = \{a \in K | n \pm a \in P \text{ for some } n \in \mathbf{N}\}$ is a valuation-ring [5]. Furthermore P can be constructed from an archimedean ordering of type 1 of the residue-class-field by means of a certain character of the value-group of $\mathfrak{o}(P)$.

3. Sums of 2^nth powers. The starting points for the applications to sums of 2^nth powers are the result $Q_n = \bigcap P$ and the facts about $\mathfrak{o}(P)$ just mentioned. Let K be an infinite not necessarily real field, $n \in \mathbf{N}$.

THEOREM 3. *If* $-1 \in Q_1$ *(i.e. K is not real), then* $-1 \in Q_n$.

This was also proved by Joly [7].

THEOREM 4. *The following statements are equivalent*: (i) *any valuation-ring of K with a real residue-class-field has a 2-divisible value-group*, (ii) $Q_1 = Q_n$ *for some n*, (iii) $Q_1 = Q_n$ *for all n*.

Theorem 4 applies to number fields, more generally to algebraic extensions of a real field with a single ordering of type 1.

THEOREM 5. *For all* $x_1, \ldots, x_r \in K$, $n, m \in \mathbf{N}$, *there exist* $y_1, \ldots, y_s \in K$ *such that*

$$(x_1^{2^n} + \cdots + x_r^{2^n})^{2^m} = y_1^{2^{n+m}} + \cdots + y_s^{2^{n+m}}.$$

Theorem 5 applied to $\mathbf{Q}(X_1, \ldots, X_r)$ generalizes in a certain sense an identity of Hilbert used in his solution of the Waring-problem [6].

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