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HOMOTOPY RIGIDITY OF LINEAR ACTIONS: CHARACTERS TELL ALL

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Our aim is to present a striking rigidity phenomenom in unitary representations of compact groups. Let $U = U(n)$ be a unitary group and H a closed subgroup of U. The homogeneous space U/H is a smooth manifold with a smooth action λ of U induced by left multiplication. If α: $G \to U$ is a representation of the compact group G, then $\lambda \circ (\alpha \times 1)$: $G \times U/H \to U/H$ is an action of G on U/H, and we denote this G-structure by $(U/H, \alpha)$. Such actions of G on U/H are called *linear* actions. We shall give a complete description of the G-homotopy types of linear actions on U/H for a certain class of H. To motivate our results we shall first examine some obvious G-equivalences of linear actions.

If X is a U-space, then the set of U-maps $\mathrm{Map}_U(U/H, X)$ is in one-to-one correspondence with elements $x \in X$ such that $U_x \supset H$, where $U_x = \{u \in U | ux = x\}$ is the isotropy group of the action at x. For example, if $a \in U$ then the element aH in U/H has isotropy group aHa^{-1} and the U-map f: $U/aHa^{-1} \to U/H$ given by $f(uaHa^{-1}) = uaH$ is a U-equivalence. Indeed if H and K are closed subgroups of U then U/H and U/K are U-equivalent if and only if $K = aHa^{-1}$ for a suitable $a \in U$. Suppose α, γ: $G \to U$ are representations such that there exists an $a \in U$ such that $\gamma(g) = a\alpha(g)a^{-1}$ for all $g \in G$ (we say that γ is *similar* to α). The map k: $(U/H, \alpha) \to (U/H, \gamma)$ given by $k(uH) = auH$ is a G-equivalence. Indeed, k is the composition of the G-equivalence $(U/H, \alpha) \to (U/aHa^{-1}, \gamma)$ induced by conjugation with a in U and the U-equivalence (hence G-equivalence!) f: $(U/aHa^{-1}, \gamma) \to (U/H, \gamma)$. Thus similarity of representations gives us G-equivalences of the associated linear actions on U/H. Here is another obvious way of obtaining G-equivalences: let c: $U \to U$ be conjugation by unitary matrices $c(a) = \bar{a}$; then if $c(H) = H$, we obtain a G-equivalence c: $(U/H, \alpha) \to (U/H, \bar{\alpha})$ where $\bar{\alpha} = c \circ \alpha$ is the representation conjugate to α.

It is too much to hope that $(U/H, \alpha)$ is G-homotopy equivalent to $(U/H, \beta)$ if and only if β or $\bar{\beta}$ is similar to α. For example, if H is a subgroup of maximal rank in U and C is the center of U then $C \subset H$ and C acts trivially on U/H, so if we let $P(U) = U/C$ be the projective unitary group (with q: $U \to P(U)$ the quotient map), then the standard left action λ of U on U/H induces an action of $P(U)$ on U/H, and it is the similarity class of the projective representation $q \circ \alpha$: $G \to P(U)$ which matters. We have: if α, β: $G \to U$ are representations and χ: $G \to S^1 = C$ is a homomorphism such that β or $\bar{\beta}$ is similar to $\chi\alpha$ then $(U/H, \alpha)$ is G-equiva-

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lent to $(U/H, \beta)$, and indeed through a map which is induced by an R-linear map of R^{2n} (the underlying real vector space of the complex vector space C^n on which $U = U(n)$ acts in the standard way). The reader would expect to find more G-equivalences of linear actions if we drop linearity, and yet more G-homotopy equivalences. The surprise is that if we make a mild restriction on H then we find that linear actions of G on U/H are *rigid under homotopy*: $(U/H, \alpha)$ is G-homotopy equivalent to $(U/H, \beta)$ if and only if they are G-equivalent through an R-linear map. Here is a sample result:

THEOREM 1 (HOMOTOPY RIGIDITY OF LINEAR ACTIONS). *If H is a subgroup of $U = U(n)$ conjugate to $U(n-k) \times T^k$, where T^k is the k-torus and $n \geqslant 2k$, α, β: $G \to U$ representations of a compact group G, then a G-map f: $(U/H, \alpha) \to (U/H, \beta)$ exists with f: $U/H \to U/H$ a homotopy equivalence if and only if there is a linear character χ: $G \to S^1$ and β or $\bar{\beta}$ is similar to $\chi\alpha$.*

We should point out that the condition $n \geqslant 2k$ is not necessary: for example, homotopy rigidity of linear actions holds for $U(5)/U(2) \times T^3$ and for $U(6)/U(2) \times T^4$, but the proof is much more involved. Similarly, the condition that H be conjugate to $U(n-k) \times T^k$ is too strong: in [13] we show homotopy rigidity of linear actions on $U(m+n+1)/U(m) \times U(n) \times U(1)$ for $mn \geqslant m+n+1$. The right level of generality for our current approach seems to be the following: let us call a subgroup H of U *friendly* if H is closed, connected, of maximal rank in $U = U(n)$ and there exists a nonzero vector $v \in C^n$ such that $hv = \lambda(h)v$ for some linear character λ: $h \to S^1$; indeed we assume H is conjugate to a subgroup $U(n_1) \times \cdots \times U(n_k) \subset U(n)$ with $n_1 \geqslant \cdots \geqslant n_k = 1$ and $n_1 + \cdots + n_k = n$ (see Borel and Siebenthal [7]). We shall outline a strategy for proving

CONJECTURE A. If H is a friendly subgroup of U then linear actions of a compact group G on U/H are rigid under homotopy.

Indeed one can conjecture that linear actions of G are rigid for U/H where H is connected of maximal rank. This is work in progress with Wu-Yi Hsiang.

An immediate consequence of our homotopy rigidity result is that the G-homotopy type of $(U/H, \alpha)$ can be read off from the character table of G (characters tell all). For example, if α, β: $G \to U$ are representations and $|\text{Trace}\, \alpha(g)| \neq |\text{Trace}\, \beta(g)|$ for some element $g \in G$, then $(U/H, \alpha)$ and $(U/H, \beta)$ have distinct G-homotopy types. An example of such a situation is given by the alternating group on five letters A_5: let α and β be the distinct irreducible 3-dimensional unitary representations, $g = (12345)$, then

$$\operatorname{Tr} \alpha(g) = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \operatorname{Tr} \beta(g) = \frac{1-\sqrt{5}}{2},$$

so $(U/H, \alpha)$ and $(U/H, \beta)$ are not A_5-homotopy equivalent for any friendly subgroup H of $U = U(3)$. Here, of course, there are no nontrivial linear characters and all characters of A_5 take real values, so two linear actions $(U/H, \gamma)$, $(U/H, \delta)$ of A_5 on U/H (with H a friendly subgroup of U) are A_5-homotopy equivalent if and only if γ is similar to δ. The case of α and β is especially interesting since there is an outer automorphism φ: $A_5 \to A_5$ with $\varphi^*\alpha = \beta$. Even the cyclic group of order two $G = Z/2Z$ gives entertaining examples: if we let 1 denote the trivial representation of G then there exist linear actions $\alpha, \beta, \gamma, \delta$ on CP^n such that $(CP^n, \alpha) \approx (CP^n, \beta)$ but $(CP^n, \alpha$

$+ 1) \not\approx (CP^n, \beta + 1)$, and $(CP^n, \gamma + 1) \approx (CP^n, \delta + 1)$ but $(CP^n, \gamma) \not\approx (CP^n, \delta)$, where we have used $\approx$ to indicate $Z/2Z$-homotopy equivalence.

In Theorem 1, f is not assumed to be a G-homotopy equivalence, that is, although there is a homotopy inverse $f'\colon U/H \to U/H$, we are not saying that such an f' can be found which is a G-map $f'\colon (U/H, \beta) \to (U/H, \alpha)$. Indeed, Petrie [15] exhibits a G-space Y, a linear action γ on $U/H = CP^k$ and a G-map $h\colon Y \to (CP^k, \gamma)$ which is a homotopy equivalence such that the induced map is equivariant K-theory

$$h^!\colon K_G(CP^k, \gamma) \to K_G(Y)$$

is not an isomorphism–this means that although h is a homotopy equivalence it is *not* a G-homotopy equivalence. Our approach is based on the fact that this sort of pathology cannot occur if Y is a complex projective space with a linear action (see [11]): given a G-map $h\colon (CP^n, \alpha) \to (CP^n, \beta)$ such that $h\colon CP^n \to CP^n$ is a homotopy equivalence, there exists an R-linear G-equivalence $k\colon (CP^n, \alpha) \to (CP^n, \beta)$ such that $h^! = k^!$ (so, in particular, $h^!$ is an isomorphism).

This report is organized as follows: in the second section, we present an exact sequence on Picard groups of G-line bundles and popularize some work of Graeme Segal [19] on cohomology of topological groups. In the third section we examine the case $U/H = CP^n$ and show how equivariant K-theory allows us to prove the homotopy rigidity theorem for this case. We also examine the general case of H a friendly subgroup of U and show how a result on cohomology automorphisms of U/H implies the homotopy rigidity theorem. The fourth section is devoted to proving the result on automorphisms of $H^*(U/H, Z)$, where H is as in Theorem 1.

A few words about the background of the problem. There is an extensive literature about G-maps of spheres with linear action: de Rham [16], Atiyah and Tall [5], Lee and Wasserman [10], Meyerhoff and Petrie [14]. The current project is the result of numerous consultations with Ted Petrie. Thanks also go to J. F. Adams, J. Dupont, H. Glover, W.-Y. Hsiang, P. Landrock, I. Madsen, G. Segal, R. Stong and J. Tornehave for their helpful comments.

2. An exact sequence of Picard groups. Let X be a G-space, $\mathrm{Pic}_G(X)$ the set of isomorphism classes of complex G-line bundles over X. We give $\mathrm{Pic}_G(X)$ the structure of a group by using the tensor product of line bundles as multiplication. If X is a CW complex, then $H^1(X;Z) \cong [X, S^1]$ and

$$H^2(X; Z) \cong [X, CP^\infty] \cong \mathrm{Pic}_E(X),$$

where $E \subset G$ is the subgroup consisting of the identity element.

THEOREM 2. *If X is a nonempty connected G-space and $H^1(X\colon Z) = 0$ then the following sequence is exact*:

$$\mathrm{Pic}_G(*) \xrightarrow{c^!} \mathrm{Pic}_G(X) \xrightarrow{i^!} \mathrm{Pic}_E(X),$$

where $c\colon X \to *$ is the collapsing map onto a point, $i\colon E \subset G$ the inclusion of the identity subgroup.

PROOF. We shall use the technique of Segal's cohomology of groups [19]: if A is an abelian G-group (G compact, A has the compactly generated topology) then cohomology groups $H^i_G(A)$ are defined for all $i \geqslant 0$. The group $H^1_G(A)$ is the quotient of the group of all crossed homomorphisms φ: $G \to A$ (functions which satisfy $\varphi(gg') = \varphi(g) + g \cdot \varphi(g')$ for all g, g' in G) modulo principal crossed homomorphisms (those which have the form $\varphi(g) = g \cdot a - a$ for a fixed a in A). The pleasant thing about Segal's cohomology is that a short exact sequence $0 \to A' \to A \to A'' \to 0$ (meaning that A is a principal A'-bundle with A'' as base) produces a long exact sequence

$$\ldots \to H^i_G(A) \to H^i_G(A'') \xrightarrow{\delta} H^{i+1}_G(A') \to H^{i+1}_G(A) \to \cdots$$

If V is a vector space over R then $H^i_G(V) = 0$ for all $i > 0$. Given our CW space X we first notice that $H^1_G(\mathrm{Map}(X, S^1))$ is precisely the set of isomorphism classes of $G \times S^1$-structures on the projection π_1: $X \times S^1 \to X$, that is, $H^1_G(\mathrm{Map}(X, S^1)) = \operatorname{Ker} i^!$. Since $H^1(X: Z) = 0$ we obtain an exact sequence

$$0 \to \mathrm{Map}(X, Z) \to \mathrm{Map}(X, R) \to \mathrm{Map}(X, S^1) \to 1,$$

$\mathrm{Map}(X, Z) = Z$ since X is connected, and the collapsing map c: $X \to *$ induces a map of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & R & \longrightarrow & S^1 & \longrightarrow & 1 \\
 & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & \mathrm{Map}(X, R) & \longrightarrow & \mathrm{Map}(X, S^1) & \longrightarrow & 1
\end{array}$$

which in turn induces maps of long exact sequences of cohomology groups. We have

$$\begin{array}{ccccccc}
H^1_G(R) & \longrightarrow & H^1_G(S^1) & \xrightarrow{\ \delta\ } & H^2_G(Z) & \longrightarrow & H^2_G(R) \\
\big\downarrow & & \big\downarrow{c^*} & & \big\downarrow{=} & & \big\downarrow \\
H^1_G(V) & \longrightarrow & H^1_G(\mathrm{Map}(X, S^1)) & \xrightarrow{\ \delta'\ } & H^2_G(Z) & \longrightarrow & H^2_G(V)
\end{array}$$

where $V = \mathrm{Map}(X, R)$ is a vector space over R, so in both exact sequences the extreme terms are zero, hence δ and δ' are isomorphisms; thus c^*: $H^1_G(S^1) \to H^1_G(\mathrm{Map}(X, S^1)) = \operatorname{Ker} i^!$ is an isomorphism, but $H^1_G(S^1) \cong \mathrm{Pic}_G(*) \cong \mathrm{Hom}(G, S^1)$, and under the isomorphism c^* corresponds to $c^!$, so Theorem 2 is proved.

Notice that $\mathrm{Pic}_E(Y) \cong H^2(Y; Z)$ under the isomorphism which assigns to a line bundle λ its first Chern class $c_1(\lambda)$.

COROLLARY 3. *Let f: $X \to Y$ be a G-map, X connected, $H^1(X; Z) = 0$, s a G-line bundle over X, t a G-line bundle over Y. Suppose $f^* c_1(i^! t) = c_1(i^! s)$, then there exists a homomorphism χ: $G \to S^1$ such that $f^! t = \chi s$.*

PROOF. Contemplate $s^{-1} \cdot f^! t$. We have

$$c_1\big(i^!(s^{-1} \cdot f^! t)\big) = -c_1(i^! s) + c_1\big(i^! f^! t\big) = -c_1(i^! s) + f^* c_1(i^! t) = 0,$$

so $s^{-1} \cdot f^! t$ is in the kernel of $i^!$, hence in the image of $c^!$–there exists a linear character χ: $G \to S^1$ with $c^! \chi = \chi \cdot 1 = s^{-1} \cdot f^! t$, or $f^! t = \chi s$, as claimed.

3. The strategy of proof. Let CP^{n-1} be a complex projective $(n-1)$-dimensional space, s: $S^{2n-1} \to CP^{n-1}$ the Hopf bundle over CP^{n-1}. If γ: $G \to U = U(n)$ is a representation, $s = s(\gamma)$: $(S^{2n-1}, \gamma) \to (CP^{n-1}, \gamma)$ defines an element in $\mathrm{Pic}_G(CP^{n-1}, \gamma)$, hence an element in $K_G(CP^{n-1}, \gamma)$ which we still call s. Let $R(G) = K_G(*)$ be the complex representation ring of G, then [3], [18] $K_G(CP^{n-1}, \gamma)$ is a free $R(G)$-module with $1, \ldots, s^{n-1}$ as basis and

$$s^n - \gamma s^{n-1} + (\Lambda^2\gamma)s^{n-2} - \cdots + (-1)^n\Lambda^n\gamma = 0,$$

where $\Lambda^i\gamma$ denotes the ith exterior power of γ.

PROPOSITION 4. *Let φ: $K_G(CP^{n-1}, \beta) \to K_G(CP^{n-1}, \alpha)$ be a homomorphism of $R(G)$-algebras with $\varphi(s(\beta)) = \chi s(\alpha)$ for some linear character χ: $G \to S^1$. Then β is similar to $\chi\alpha$.*

PROOF. Let $s(\alpha) = s$, $s(\beta) = t$. Then t satisfies

$$t^n - \beta t^{n-1} + \cdots + (-1)^n\Lambda^n\beta = 0.$$

Hence applying φ we have

$$\chi^n s^n - \beta\chi^{n-1}s^{n-1} + \cdots + (-1)^n\Lambda^n\beta = 0,$$

and multiplying with χ^{-n} we obtain

$$s^n - \beta\chi^{-1}s^{n-1} + \cdots + (-1)^n\Lambda^n(\beta\chi^{-1}) = 0.$$

But

$$s^n - \alpha s^{n-1} + \cdots + (-1)^n\Lambda^n\alpha = 0$$

and $K_G(CP^{n-1}, \alpha)$ is $R(G)$-free on $1, \ldots, s^{n-1}$. Comparing the coefficients of s^{n-1} we obtain $\beta\chi^{-1} = \alpha$ in $R(G)$ as claimed.

We shall now show how homotopy rigidity of linear actions on CP^{n-1} follows (compare [11], [12]). Let f: $(CP^{n-1}, \alpha) \to (CP^{n-1}, \beta)$ be a G-map so that f^*: $H^*(CP^{n-1}; Z) \to H^*(CP^{n-1}; Z)$ is an isomorphism. Let $u = c_1(s) = c_1(i^!s(\alpha))$, the first Chern class of the Hopf bundle s. Then $f^*u = u$ or $-u$ since f^* is an isomorphism and $H^2(CP^{n-1}; Z)$ is generated by u. If $f^*u = -u$, we replace f by $c \circ f$ and β by $\bar{\beta}$ (where c: $CP^{n-1} \to CP^{n-1}$ is induced by conjugation in $U = U(n)$), so we may assume $f^*u = u$, that is $f^*c_1(i^!t) = c_1(i^!s)$. We apply Corollary 3: there exists a linear character χ: $G \to S^1$ such that $f^!t = \chi s$. Applying Proposition 4 to $\varphi = f^!$ we obtain that β is similar to $\chi\alpha$. Recalling that we may have had to replace our original β by $\bar{\beta}$ to obtain $f^*u = u$ we obtain the homotopy rigidity result for linear actions on CP^{n-1}.

We build our approach to linear actions on U/H on this special case of CP^{n-1}. Suppose H is a friendly subgroup of $U = U(n)$; there exists a nonzero vector $v \in C^n$ such that $hv = \lambda(h)v$ for all $h \in H$ for some linear character λ. We define a map π: $U/H \to CP^{n-1}$ by $\pi(uH) = [uv]$. If α: $G \to U$ is a representation then π is a G-map π_α: $(U/H, \alpha) \to (CP^{n-1}, \alpha)$.

PROPOSITION 5. *If H is a friendly subgroup of $U = U(n)$ and π_α is as above, then $\pi_\alpha^!$: $K_G(CP^{n-1}, \alpha) \to K_G(U/H, \alpha)$ is a monomorphism.*

PROOF. We may as well assume $H = U(n_1) \times U(n_2) \times \cdots \times U(n_k)$ with $n_k = 1$ and $v = \varepsilon_n$, the last vector in the standard basis of C^n, then π is induced by the inclusion $H \subset U(n-1) \times U(1)$. Let $T = U(1) \times \cdots \times U(1)$ be the standard n-torus of U consisting of diagonal matrices, then $T \subset H \subset U$ induces a commutative diagram of projections

$$\begin{array}{ccc} (U/T, \alpha) & \longrightarrow & (U/H, \alpha) \\ & \underset{\rho}{\searrow} & \downarrow \pi_\alpha \\ & & (CP^{n-1}, \alpha) \end{array}$$

and since $\rho^!$ is a monomorphism (see [18]), so is $\pi_\alpha^!$.

Now let α, β: $G \to U$ be representations, $s = s(\alpha)$, $t = s(\beta)$ the G-Hopf bundles on (CP^{n-1}, α) and (CP^{n-1}, β), respectively. Let f: $(U/H, \alpha) \to (U/H, \beta)$ be a G-map such that f: $U/H \to U/H$ is a homotopy equivalence. Let $u = c_1(i^! \pi_\alpha^! s) = c_1(i^! \pi_\beta^! t)$. If $f^* u = u$, then as before Corollary 3 says that there exists a linear character χ: $G \to S^1$ such that $f^! \pi_\beta^! t = \chi \pi_\alpha^! s = \pi_\alpha^!(\chi s)$. Thus $f^!$ maps the image of $\pi_\beta^!$ into the image of $\pi_\alpha^!$. Since $\pi_\alpha^!$ is a monomorphism, we may define

$$\varphi = \left(\pi_\alpha^!\right)^{-1} f^! \pi_\beta^!: K_G(CP^{n-1}, \beta) \to K_G(CP^{n-1}, \alpha)$$

which, of course, is a map of $R(G)$-algebras and $\varphi(t) = \chi s$, so Proposition 4 says that β is similar to $\chi\alpha$. The catch, of course, is that there is no reason to expect that $f^* u$ is equal to u, so we have to do more work.

The group of U-maps $\mathrm{Map}_U(U/H, U/H)$ is isomorphic to $N_U(H)/H$, where $N_U(H) = \{a \in U \mid aHa^{-1} = H\}$ is the normalizer of H in U (see Bredon [8], Samelson [17]). If γ: $G \to U$ is a representation and k: $U/H \to U/H$ is a U-map, then k: $(U/H, \gamma) \to (U/H, \gamma)$ is a G-map. Let c: $U \to U$ be given by $c(u) = \bar{u}$, the matrix with complex conjugate entries. We have chosen H in its conjugacy class so that $c(H) = H$, hence c: $(U/H, \gamma) \to (U/H, \bar{\gamma})$ is a G-map. We have a homomorphism

$$\psi: N_U(H)/H \times Z/2Z \to \mathrm{Aut}(H^*(U/H; Z))$$

given by $\psi(k, t) = k^* \circ c^{t*}$. Stated in another way: the homomorphism ψ defines an action of $N_U(H)/H \times Z/2Z$ on $H^*(U/H; Z)$. Of course the group $\mathrm{Homeq}(U/H)$ of all homotopy classes of homotopy equivalences of U/H also acts on $H^*(U/H; Z)$ by taking induced homomorphisms in cohomology. We now state several related conjectures.

CONJECTURE B. Let H be a friendly subgroup of $U = U(n)$, π: $U/H \to CP^{n-1}$ the standard map, $u = \pi^* c_1(s)$, where s is the Hopf bundle on CP^{n-1}, then the orbit of u under $N_U(H)/H \times Z/2Z$ is the same as the orbit of u under $\mathrm{Homeq}(U/H)$.

PROPOSITION 6. *Conjecture* B *implies Theorem* 1 (*homotopy rigidity of linear actions on* U/H).

PROOF. We keep the notation of our earlier discussion: let α, β: $G \to U = U(n)$ be representations, f: $(U/H, \alpha) \to (U/H, \beta)$ a G-map such that f:

$U/H \to U/H$ is a homotopy equivalence, $u = \pi^* c_1(s)$. According to Conjecture B there exists an element k of $N_U(H)/H \times Z/2Z$ such that $k^* f^* u = u$. Replace f by $f \circ k$ (here we may have to replace α by $\bar{\alpha}$ if conjugation is involved). Then $f^* u = u$, so $i^! f^! \pi_\beta^! t = i^! \pi_\alpha^! s$, and since U/H is connected and simply connected, we obtain from Corollary 3 a linear character χ: $G \to S^1$ such that $f^! \pi_\beta^! t = \chi \pi_\alpha^!(s)$. So now letting $\varphi = \pi_\alpha^{!-1} f^! \pi_\beta^!$ we can apply Proposition 4 to conclude that β is similar to $\chi\alpha$.

We shall prove an even stronger result for a multitude of subgroups of U:

CONJECTURE C. The map ψ is an isomorphsm of $N_U(H)/H \times Z/2Z$ onto the group of all algebra isomorphisms of $H^*(U/H; Z)$ if H is a friendly subgroup of $U = U(n)$ and $n \geqslant 3$.

Notice that $U(2)/T^2 \approx S^2$, and in this case ψ has a cyclic group or order 2 as a kernel. If $n \geqslant 3$, ψ is a monomorphism.

Let us boldly walk even further on the limb: the following algebraic conjecture implies Conjecture B (and Conjecture C in a lot of cases).

CONJECTURE D. Let T be the standard torus of $U = U(n)$, $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ the standard basis for C^n, let π_i: $U/T \to CP^{n-1}$ for $i = 1, \dots, n$ be given by $\pi_i(uT) = [u\varepsilon_i]$, s the Hopf bundle on CP^{n-1}, let $x_i = \pi_i^* c_1(s)$. If $x \in H^2(U/T; Z)$ and $x^n = 0$ then there exists an integer a and an i in $\{1, 2, \dots, n\}$ such that $x = ax_i$.

The algebraic data are easy to state: $H^*(U/T; Z) = Z[x_1, \dots, x_{n-1}]$ modulo the ideal $I_n = (h_2, \dots, h_n)$, where h_i is the sum of all monomials of degree i in $x_1, \dots, x_{n-1}$ (see Borel [6])–for example, for $n = 4$, $h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$. It is important to notice that $n - 1$ appears above, not n–indeed $x_n = -x_1 - x_2 - \cdots - x_{n-1}$. The group $N_U(T)/T$ is S_n, the symmetric group on n letters which acts on $H^*(U/T; Z)$ by permuting the $x_1, \dots, x_n$. Conjecture D is trivial to prove for $n = 3$. For $n = 4$, the algebra is already delightfully complicated and a hint is helpful: examine the solutions of $x^4 = 0$ first over $Z/3Z$ and then exploit the fact that multiplication by x_1 from $H^6(U(4)/T^4; Z/3Z)$ to $H^8(U(4)/T^4; Z/3Z)$ has kernel of dimension one to show that if $x_1 + y$ is a solution of $x^4 = 0$ over Z and $y = bx_2 + cx_3$ then for all natural numbers k we have $3^k | y$ implies $3^{k+1} | y$, so $y = 0$.

The limb is beginning to creak ominously, but let's take one more step:

CONJECTURE E. If H is a connected subgroup of maximal rank of $U = U(n)$ and Homeq(U/H) is the group of homotopy classes of homotopy equivalences of U/H then $N_U(H)/H \times Z/2Z$ is a normal subgroup of Homeq(U/H) if $n \geqslant 3$.

One reason for thinking wishfully about Conjecture E is that it would give a beautifully simple proof of Conjecture C for $H = T$, the maximal torus of $U(n)$ and Homeq(U/H) is the group of homotopy classes of homotopy equivalences of U/H then $N_U(H)/H \times Z/2Z$ is a normal subgroup of Homeq(U/H) if $n \geqslant 3$.

4. Algebra automorphisms of $H^*(U/H; Z)$. We shall prove Conjecture C for $U = U(n)$, $H = U(n-k) \times T^k$, $n \geqslant \max\{2k, k+2\}$. As before, let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ be the standard basis of C^n and let π_i: $U/H \to CP^{n-1}$ be the projection $\pi_i(uH) = [u\varepsilon_{n-k+i}]$ for $i = 1, \dots, k$. Let $y \in H^2(CP^{n-1}; Z)$ be the Chern class of the Hopf bundle and let $x_i = \pi_i^*(y)$; then $H^*(U/H;$

$Z) = Z[x_1, \dots, x_k]/I$, where the ideal $I = (h_{n-k+1}, \dots, h_n)$ and h_j is the sum of all monomials of degree j in $x_1, \dots, x_k$. A free basis for H^* is given by $x^E = x_1^{e_1}x_2^{e_2}\dots x_k^{e_k}$, where $0 \leqslant e_i < n - k + i$ (see Borel [6]). The group $N_U(H)/H$ is S_k, the symmetric group on k letters, and it acts on $H^*(U/H;$ $Z)$ by permuting $x_1, \dots, x_k$. We examine the case of $k = 2$ more closely.

LEMMA 7. *If $u = ax_1 + bx_2$ is an element in $H^2(U(m+2)/U(m) \times T^2;$ $Z)$ with $u^{2m} = 0$, then either $a = 0$ or $b = 0$.*

PROOF. We first claim that if both a and b are nonzero and $u^{2m+1} = 0$ then $a = b$. Notice that $x_1^{m+2} = x_2^{m+2} = 0$ (since both come from CP^{m+1}) and H^{4m+2} has $x_1^m x_2^{m+1}$ as basis. Moreover, $x_1^{m+1}x_2^m = -x_1^m x_2^{m+1}$. We have

$$\begin{aligned} 0 = u^{2m+1} &= (ax_1 + bx_2)^{2m+1} \\ &= \binom{2m+1}{m+1} a^{m+1} b^m x_1^{m+1} x_2^m + \binom{2m+1}{m} a^m b^{m+1} x_1^m x_2^{m+1}, \end{aligned}$$

so $a \neq 0$, $b \neq 0$ implies $a = b$. If now, in addition, $u^{2m} = 0$, then we have

$$\begin{aligned} 0 = \binom{2m}{m+1} a^{2m} x_1^{m+1} x_2^{m-1} &+ \binom{2m}{m} a^{2m} x_1^m x_2^m \\ &+ \binom{2m}{m-1} a^{2m} x_1^{m-1} x_2^{m+1}, \end{aligned}$$

but H^{4m} has $\{x_1^m x_2^m, x_1^{m-1}x_2^{m+1}\}$ as basis and

$$x_1^{m+1}x_2^{m-1} = -x_1^m x_2^m - x_1^{m-1}x_2^{m+1},$$

so the above sum reduces to

$$0 = \left\{ \binom{2m}{m} - \binom{2m}{m+1} \right\} a^{2m} x_1^m x_2^m,$$

so since $\binom{2m}{m} \neq \binom{2m}{m+1}$ for all m it follows that $a = 0$, a contradiction to our temporary hypothesis that $a \neq 0$ and $b \neq 0$.

COROLLARY 8. *Let $v \in H^2(U(m+k)/U(m) \times T^k; Z)$ be an element such that $v^{m+k} = 0$. If $m \geqslant k$ then $v = ax_i$ for some i in $\{1, \dots, k\}$.*

PROOF. By applying a suitable element of S_k we can assume that the coefficient of x_1 is nonzero. We wish to show that the coefficient of x_i for $i \neq 1$ is zero–and, of course, it is sufficient to prove this for $i = 2$. Consider the standard map

$$j\colon U(m+2)/U(m) \times T^2 \to U(m+k)/U(m) \times T^k$$

induced by the standard inclusion $C^{m+2} \subset C^{m+k}$ under which $j^*x_1 = x_1$, $j^*x_2 = x_2, j^*x_i = 0$ for $i > 2$. Inspect $u = j^*v$; then $u^{m+k} = 0, m + k \leqslant 2m$; hence $a \neq 0$ implies that the coefficient of x_2 is zero.

We are now ready to prove Conjecture C for $U(m+k)/U(m) \times T^k$.

THEOREM 9. *If $n \geqslant \max\{2k, k+2\}$, $U = U(n)$, $H = U(n-k) \times T$, then the map ψ is an isomorphism of $N_U(H)/H \times Z/2Z$ onto the group of all algebra isomorphisms of $H^*(U/H; Z)$.*

PROOF. We first prove that ψ is onto. Let $\varphi\colon H^*(U/H; Z) \to H^*(U/H; Z)$