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A FUBINI THEOREM FOR ITERATED STOCHASTIC INTEGRALS

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Let $x(t)$ and $y(t)$ be continuous martingales on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Consider a bounded region $\mathcal{D}$ in $\mathbf{R}_+^2$ with smooth boundary Γ. Define Γ_i to be the portion of Γ along which the outward normal points in the direction of quadrant i $(i = 1, 2, 3, 4)$. Let $f(t, s)$ be a smooth function on $\overline{\mathcal{D}}$. It is desired to evaluate

$$\iint_{\mathcal{D}} f(t, s)\, dx(s)dy(t) \quad \text{and} \quad \iint_{\mathcal{D}} f(t, s)\, dy(t)dx(s)$$

by Riemann sums in an Ito-belated fashion. The difference between these sums is then of the form

$$\pm \Sigma f(t_i', s_i')[x(s_{i+1}') - x(s_i')]\,[y(t_{i+1}') - y(t_i')]$$

where (t_i', s_i') are points near Γ_2 and Γ_4. Under suitable conditions on $\mathcal{D}$ this sum tends to an integral of f along these portions of Γ. These considerations lead to the following

THEOREM (THE CORRECTION FORMULA). *Let $(x(t), y(t))$ be a joint martingale. Then*

$$\iint_{\mathcal{D}} f(t, s)\, dx(s)dy(t) + \int_{\Gamma_2 \cap l} f(t, t)\, d\langle x, y\rangle(t)$$
$$= \iint_{\mathcal{D}} f(t, s)\, dy(t)dx(s) + \int_{\Gamma_4 \cap l} f(t, t)\, d\langle x, y\rangle(t)$$

where $\langle x, y\rangle$ is the quadratic covariation process of x and y, and l is the line $s = t$.

It is to be noted that the rigorous justification of the Correction Formula entails the development of a new type of stochastic integral $I(t) = \int_{t_0}^t g(t, s)\, dx(s)$ where $g(t, s)$ is measurable with respect to the sigma-field generated by $\{x(u) - x(s)\colon s \leqslant u \leqslant t\}$. Conditions are given which insure the existence of $I(t)$ as a limit of Ito-belated Riemann sums. The following result concerning the moments of $I(t)$ is presented.

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THEOREM. *Suppose* $\mathbf{E}(g(t, s)/F_s) = 0$, *a.s. where* F_s *is the sigma-field generated by* $\{x(u): t_0 \leqslant u \leqslant s\}$. *Then*

$$\mathbf{E}I(t) = -\int_{t_0}^{t} \mathbf{E}\partial_s g(t, s)\, dx(s)$$

$$\mathbf{E}|I(t)|^2 = \int_{t_0}^{t} \mathbf{E}|g(t, s)|^2\, ds + \int_{t_0}^{t}\int_{t_0}^{t} \mathbf{E}\partial_s g(t, s)\partial_r g(t, r)\, dx(s)dx(r)$$

where ∂_s *represents the s-stochastic differential.*

As an application of the Correction Formula, consider the linear Ito-Volterra equation (see Berger [1])

$$\xi(t) - \int_{t_0}^{t} a(t, s)\xi(s)\, dw(s) - \int_{t_0}^{t} b(t, s)\xi(s)\, ds = F(t).$$

Here $w(t)$ is a Brownian motion. Using the Correction Formula the solution can be obtained by a Neumann series, and takes the form

$$\xi(t) = F(t) + \int_{t_0}^{t} r_a(t, s)F(s)\, dv(s) + \int_{t_0}^{t} r_b(t, s)F(s)\, ds$$

where $v(t) = w(t) - \int_{t_0}^{t} a(s, s)\, ds$ and

$$r_a(t, s) = \sum_{n=1}^{\infty} a_n(t, s), \qquad r_b(t, s) = \sum_{n=1}^{\infty} b_n(t, s),$$

$$a_1(t, s) = a(t, s), \qquad b_1(t, s) = b(t, s),$$

$$a_{n+1}(t, s) = \int_{s}^{t} a(t, r)a_n(r, s)\, dw(r) + \int_{s}^{t} b(t, r)a_n(r, s)\, dr,$$

$$b_{n+1}(t, s) = \int_{s}^{t} a(t, r)b_n(r, s)\, dw(r) + \int_{s}^{t} b(t, r)b_n(r, s)\, dr; \qquad n = 1, 2, \ldots .$$

Corollaries of the Correction Formula include a differentiation rule for $z(t) = h(t, x(t))$ where $h(t, a) = \int_{t_0}^{t} \psi(s, x(s), t, a)\, dx(s)$; and a formula for evaluating $\int_{t_0}^{t} \varphi(s)H_n(\int_s^t \varphi^2(r)\, dr, \int_s^t \varphi(r)\, dw(r))\, dw(s)$ where $H_n(t, x)$ is the Hermite polynomial of degree n. There is also included a discussion of stochastic integrals $\int_{t_0}^{t} x(\lambda(s))\, dx(s)$ where $\lambda(s) \geqslant s$.

REFERENCES

1. M. A. Berger, *Stochastic Ito-Volterra equations,* Ph.D. Thesis, Pittsburgh, Carnegie-Mellon University, 1977.

2. P. A. Meyer, *Seminaire de Probabilities.* X, Lecture Notes in Math., vol. 511, Springer-Verlag, New York, 1976, pp. 321–326.

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