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ON LIE ALGEBRAS OF DIFFERENTIAL FORMAL GROUPS OF RITT

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Between February 1949 and August 1950 J. F. Ritt published four papers [8]–[11] about differential formal groups. This research was stopped by his death on January 5, 1951. Below we describe some results which can be considered as a continuation of the research begun in the abovementioned papers of Ritt.

1. Let K be an algebraically closed field of characteristic 0 and let k be a subfield of K. A finite-dimensional vector K-space D is called (after W. Y. Sit) a Lie K-space if D has a structure of a Lie k-algebra. Suppose now that $D \subseteq \mathrm{Der}_k K$ and that $k = \{a \in K | Da = 0\}$. We denote by $K[D]$ the (associative) algebra of differential operators; it is generated by K and D with relations $d\lambda = \lambda d + d(\lambda)$, $d_1d_2 - d_2d_1 = [d_1, d_2]$ for $d, d_1, d_2 \in D$, $\lambda \in K$. If M, N are $K[D]$-modules then $M \otimes_K N$ and $\mathrm{Hom}_K(M, N)$ are given natural structures of $K[D]$-modules by $d(m \otimes n) = dm \otimes n + m \otimes dn$ and $(d\varphi)(m) = d(\varphi(m)) + \varphi(-dm)$ for $\varphi \in \mathrm{Hom}_K(M, N)$, $m \in M$, $n \in N$, $d \in D$.

1.1 DEFINITION. A $K[D]$-algebra A is a map $A \otimes_K A \longrightarrow A$ of $K[D]$-modules. A $K[D]$-coalgebra C is a map $C \longrightarrow C \otimes_K C$ of $K[D]$-modules. A $K[D]$-bialgebra B is a $K[D]$-algebra together with a $K[D]$-coalgebra structure $B \longrightarrow B \otimes_K B$ which is a map of $K[D]$-algebras.

1.2 DEFINITION. A $K[D]$-module M is split if $M = K \otimes_k M^0$ for some k-module M^0 and $DM^0 = 0$. The split action of $d \in D$ will be denoted d^0.

1.3 REMARK. If M is a split $K[D]$-module and we have another action of the elements $d \in D$ on M which makes M into a $K[D]$-module, then $d - d^0 \in \mathrm{End}_K M$. For $\omega(d) = d - d^0$ we have the relation

$$\omega([d_1, d_2]) = d_1^0\omega(d_2) - d_2^0\omega(d_1) + [\omega(d_1), \omega(d_2)].$$

Therefore $\omega(d)$ can be considered as a differential 1-form with values in the Lie K-space $\mathrm{End}_k M$, and the above relation reduces to $\delta\omega = -\tfrac{1}{2}[\omega, \omega]$, where δ denotes the exterior differentiation of differential forms, given by the formula

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$$(\delta\omega)(d_1, d_2) = \tfrac{1}{2}\omega([d_1, d_2]) - \tfrac{1}{2}d_1^0\omega(d_2) + \tfrac{1}{2}d_2^0\omega(d_1).$$

If M has a split $K[D]$-algebra structure and we have another action of the elements $d \in D$ on M which makes the K-algebra M into a $K[D]$-algebra then $\omega(d) \in \mathrm{Der}_K M$.

1.4 The group $\mathrm{Aut}_K M$ of K-module or K-algebra automorphisms of a split $K[D]$-structure on M acts on the set of all $K[D]$-structures by $(g^{-1}\omega)(d) = g^{-1} \cdot d(g) + g^{-1}\omega(d)g$. Here we denote by $d(g)$ the action of $d \in D$ on $g \in \mathrm{End}_K M = \mathrm{Hom}_K(M, M)$. Denoting by $g^{-1} \cdot \delta g$ the differential 1-form $d \longrightarrow g^{-1} \cdot d(g)$ we have $g^{-1}(\omega) = g^{-1} \cdot \delta g + g^{-1}\omega g$, i.e., the equivalence class of ω is a connection ("in a principal bundle with fiber $\mathrm{End}_K M$").

1.5 If the $K[D]$-module M is finitely generated then the $K[D]$-module M^* has the linearly compact topology [4] and $M = (M^*)^*$ where the second star denotes the topological dual.

2. **Definition.** A regular complete local commutative $K[D]$-finitely generated continuous $K[D]$-bialgebra A is called a differential formal group. (Ritt considered the case in which $\dim D = 1$ and A is the algebra of formal power series over a $K[D]$-free module.)

Let J be the maximal ideal of A (where A is as above). Then the vector space $(J/J^2)^*$ is given, as usual, the structure of a Lie algebra, called the Lie algebra of the formal group. In our case the Lie algebra will be a linearly compact $K[D]$-algebra.

3. Our aim is to state some structure results about linearly compact Lie $K[D]$-algebras G such that G^* (topological dual) is finitely generated. We denote the set of such algebras by $\mathfrak{G} = \mathfrak{G}(D)$.

3.1 PROPOSITION. *Let $G \in \mathfrak{G}$. Then G has a greatest solvable $K[D]$-ideal. It is the sum of all solvable $K[D]$-ideals and its derived series terminates at zero in a finite number of steps.*

3.2 PROPOSITION. *Let $G \in \mathfrak{G}$ be a $K[D]$-simple algebra. Then at least one of the following holds,*

(i) *G is K-simple and therefore (cf. [5]) either finite dimensional or of Cartan type W_n, S_n, H_n, K_n.*

(ii) *There exists a Lie K-subspace $\widetilde{D}$ and a K-simple Lie $K[\widetilde{D}]$-algebra $S \in \mathfrak{G}(\widetilde{D})$ such that $G = \mathrm{Hom}_{K[\widetilde{D}]}(K[D], S)$.*

The proof of (ii) in the above proposition is modelled on Blattner's proof [1], [2] of a Guillemin result [4]. The difficulty is that D acts on K and that $K[D]$ is not a bialgebra.

4. The above results show that completely to describe the simple $K[D]$-algebras $\mathfrak{G}(D)$ it is sufficient to consider only K-simple ones. To begin with we

note that such algebras always have a split $K[D]$-structure (because K is algebraically closed). As usual we denote this structure by d^0. If G is not finite dimensional and G has the split structure then G^* is not finitely generated. Let $\omega(d) = d - d^0$ as above. First we have

THEOREM. *If $G \in \mathfrak{G}(D)$ is simple and finite dimensional then for every differential* 1*-form ω satisfying $\delta\omega = -\frac{1}{2}[\omega, \omega]$, there exist an extension L of K and $g \in$* (Aut G)(L) *such that ω is an exact differential: $\omega = g^{-1} \cdot \delta g$.*

The proof is modelled on proofs of Cassidy [3] and Kovacic [7] and also was suggested by J. Tits.

5. In the case when G is of Cartan type, I was able to obtain conclusive results only in the "Euclidean" case, i.e. the case in which D has a commutative basis. (J. Hrabowski has informed me recently that this condition holds automatically.) We recall that G is graded $G = \Sigma_{i \geqslant -2} G_i$ and therefore $\mathrm{Der}_K\, G = \Sigma_{i \geqslant -2}(\mathrm{Der}_K G)_i$. Write $\omega(d) = \Sigma\omega(d)_i$, $\omega(d)_i \in (\mathrm{Der}_K G)_i$. We define $\psi_\omega: D \longrightarrow \Sigma_{i<0}(\mathrm{Der}_K G)_i$ by $\psi_\omega(d) = \Sigma_{i<0}\omega(d)_i$.

5.1 THEOREM. *Suppose that D has a commutative basis and $G \in \mathfrak{G}$. Then $\psi_\omega(D) = \Sigma_{i<0}(\mathrm{Der}_K G)_i$.*

Let $\widetilde{D} = \mathrm{Ker}\, \psi_\omega$. Then D is a Lie K-subspace of D of codimension $n = \Sigma_{i<0}\dim(\mathrm{Der}_K G)_i$. The set of all such subspaces in D we denote Γ_n. Let $\Pi(G)$ denote the set of $K[D]$-structures on G which belong to $\mathfrak{G}$. The above theorem defines a map $\pi: \Pi(G) \longrightarrow \Gamma_n$.

5.2 THEOREM. (i) *If G is of type W_n then π is a bijection of* $\Pi(G)$ *and* Γ_n.

(ii) *If G is of type S_n or H_n then the fibers of π are subsets of K^*/k^* or* $\mathrm{GL}(n, K)/\mathrm{Sp}(n, K)k^*$ *respectively.*

(iii) *If G is of type K_n then the fibers of π are subsets of some algebraic homogeneous space for* $\mathrm{GL}(n, K)$.

6. The above suggests the following definition of an affine $K[D]$-algebraic variety: it is the prime spectrum of a commutative $K[D]$-finitely generated $K[D]$-algebra. In particular, an affine $K[D]$-algebraic group is Spec A where A is a commutative $K[D]$-bialgebra. In particular, this group (considered as a K-group) is a projective limit of finite dimensional affine algebraic K-groups. This shows that the Lie algebras of Cartan type are not Lie algebras of affine $K[D]$-algebraic groups (if the above definition is accepted). Our definition differs from that of Cassidy [3].

7. The algebras studied by A. A. Kirillov in [6] are (modulo the replacement of a ring by a field and C^∞ by analyticity) our algebras with the additional condition that G^* have one generator. Kirillov's condition of transitivity implies that they are of Cartan type, so they are either K_n or the one-dimensional central extension of H_n.

8. To conclude this announcement I express my deep gratitude to V. Guillemin, D. Kazhdan and E. Kolchin who had patience to listen to me and who gave me much help.

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