by Mackey during the summer quarter at the University of Chicago. The necessarily brief lectures covering global representation theory, direct integral decompositions, and induced representations were written up by Michael Fell and David Lowdenslager. This new book consists of two parts, the first being the original Chicago notes with minor corrections and clarifications, the second being a survey of early history and a long yet necessarily incomplete sketch of developments since 1955.

Unfortunately the second part which is nearly as long as the first has no table of contents. The only way to know what it contains or to find anything is to thumb through the pages searching for section headings. This is a definite nuisance. There is another irritating peculiarity; numbers 1 through 150 of the extensive 356 item bibliography for the second part appear elsewhere in a 1963 survey article in the Bulletin. These defects should be easy to remedy. Perhaps the University of Chicago Press can be persuaded to issue recall notices?

The lengthy sketch of developments after 1955 is divided into 14 sections. To mention just a few of its topics, it treats extensions of Mackey's own work such as the imprimitivity theorem, normal subgroup analysis and so on, as well as the state of unitary representation theory for nilpotent and solvable Lie groups, harmonic analysis, connections of representation theory with number theory, and applications to physics. These topics are discussed and interpreted from the point of view of the general theory presented in the notes. Thus, harmonic analysis is viewed as being essentially the problem of decomposing group actions on function spaces and more specifically as a problem of decomposing certain induced representations; the Selberg trace formula is seen as a special case of a Plancherel formula for representations induced by representations of discrete subgroups where the quotient is compact.

One might well argue that this perspective is too narrow, i.e., that the material in the notes is either not appropriate or not adequate for a number of topics, in particular, that it is not very helpful in certain questions concerning representations of semisimple Lie groups. For example, it is perhaps desirable but unlikely that global methods such as semidirect product analysis will succeed in replacing infinitesimal arguments and the subquotient theorem of Harish-Chandra in the analysis of the irreducible representations of semisimple Lie groups. But in any case, it makes for very interesting reading, often provides new insights, and presents an important unified view of representation theory. More to the point, however, the second part of the book admirably serves to fulfill its intended purpose, namely, that of placing the original and by now classic material in the 1955 lectures in a present day context.

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Geometric algebra over local rings, by Bernard R. McDonald, Dekker, New York and Basel, 1976, xii + 421 pp., $29.50.

The name *geometric algebra* stems from E. Artin's book of that title [A] and refers to the algebra that centers around the foundations of affine and projective geometry and the geometry of quadratic forms. Artin's intent in writing his book was "to devise a course of geometric nature which . . . can be presented to beginning graduate students or even to advanced undergraduates," and he comes out strongly in the book for replacing matrix arguments by geometric arguments: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out"; cf. [**A**, pp. 13–14]. Similar sentiments are expressed by Baer in [**B**] and Dieudonné in [**D**].

In the book under review the author suppresses the role of matrices in favor of algebraic arguments involving homomorphisms of free modules to just about the right extent. At the same time, he makes little effort to explain the geometry behind these arguments; but since the book covers essentially the same ground as Artin's, this should present no great problem for the reader provided he keeps Artin's book close at hand for frequent reference. Those with a taste for having everything laid out in a neat succession of theorems and proofs might even prefer McDonald's rather formal exposition.

This book differs from Artin's in that the latter deals with finite dimensional vector spaces over a skew field whereas the former deals with finite rank free modules over a local ring. (Here "ring" means a commutative ring with identity, and "local ring" means a ring with a unique maximal ideal.) Vector spaces over a skew field arise very naturally in synthetic projective geometry: they correspond to those projective spaces, defined axiomatically, for which Desargues' theorem holds. This theorem dates back to 1639, long before the invention of skew fields. Pappus' theorem plays a similar role with respect to the possibility of assigning the points of a projective space coordinates from a field. On the other hand, I know of no such historical connections for geometric algebra over a local ring; presumably the subject arose as a natural generalization of the field case and must stand on its own merits.

The author offers in the preface some justifications for writing a book on this subject, one of these being that it provides a good introduction to the study of projective modules over an arbitrary ring and, in particular, to algebraic K-theory. I agree with the general intent of this remark, although the reader should be warned that the rings of interest in algebraic K-theory are not themselves local and that the groups K_0 and K_1 are, in fact, trivial for a local ring. Perhaps the kind of result that the author has in mind here, since he brings up the Serre conjecture in his discussion at the end of Chapter I, is Suslin's recent theorem that if $D[x]$ is the polynomial ring in one variable over a domain D and $(f) = (f_1(x), \ldots, f_n(x))$ is a unimodular n-tuple from $D[x]$ such that $f_1(x)$ is monic, then there exists an $A \in SL_n(D[x])$ such that $(f)A = (f_1(0), \ldots, f_n(0))$. This is a key step in Suslin's proof of the Serre conjecture, and although really quite a different result from the theorems of this book, it nonetheless has the ring of some of them, e.g., the theorem on p. 73 that if V is a rank n free R-module, R local, then $GL_n(R)$ acts transitively on vectors having the same order ideal. (The order ideal of an element $v \in V$ is the ideal of R generated by the coefficients of v when v is expressed in terms of a basis. The element v is called unimodular if its order ideal is R.)

To get a feeling for the generalization from the field to the local ring case, let us look at the so-called "fundamental theorem of projective geometry" (abbreviated FT). This is the main theorem of the first chapter and is vital for a number of later applications. Let R be a ring, let V be a free R-module of rank $n \geqslant 3$, and let $P(V)$ be the set of rank 1 free (direct) summands of V. Artin's version of the fundamental theorem for R a field asserts that every bijection $\sigma: P(V) \to P(V)$ such that

$$(*) \qquad l \subset l_1 + l_2 \Rightarrow \sigma(l) \subset \sigma(l_1) + \sigma(l_2) \quad \text{for all } l, l_1, l_2 \in P(V)$$

is induced by a semilinear bijection of V; and, moreover, this semilinear bijection is unique up to unit multiples. (A semilinear bijection λ_μ is a 1-1, onto map $\lambda: V \to V$, together with an automorphism μ of R, such that $\lambda(v_1 + v_2) = \lambda(v_1) + \lambda(v_2)$ and $\lambda(av) = \mu(a)\lambda(v)$, for all $v, v_1, v_2 \in V$, $a \in R$.) Such maps σ are called collineations (McDonald prefers "projectivities", which agrees with the terminology of [B] but conflicts with that of [A] and [S]). Since it is immediate that, conversely, every semilinear bijection of V induces such a collineation σ of $P(V)$, it follows that the collineations are exactly the maps induced by semilinear bijections of V.

There are (at least) two possible generalizations. One of these, valid for an arbitrary ring R, merely replaces the $\Rightarrow$ of $(*)$ by $\Leftrightarrow$; cf. [OS] (Ojanguren and Sridharan give an example to show that the statement is false for arbitrary R without this modification). The other route, chosen by McDonald, is to restrict to a local ring R and replace $(*)$ by

$$(**) \qquad \begin{array}{l} l_1 + l_2 \text{ is a rank 2 free summand of } V \Leftrightarrow \sigma(l_1) + \sigma(l_2) \text{ is a} \\ \text{rank 2 free summand, and for any such } l_1 + l_2,\ l \subset l_1 + l_2 \\ \Leftrightarrow \sigma(l) \subset \sigma(l_1) + \sigma(l_2). \end{array}$$

As McDonald points out, this latter version has the advantage that one only need check the hypothesis for rank 2 free summands rather than for all subspaces of the form $l_1 + l_2$.

The interpretation of this theorem in terms of classical projective geometry (over a field) runs as follows. $P(V)$ should be thought of as the set of points of a projective space and $(*)$ as the requirement that collinear points map into collinear points. For example, if two planes $P(V)$ and $P'(V)$ are embedded in projective 3-space and $P(V)$ is projected onto $P'(V)$ from some point not on either plane, then the resulting map, called a perspectivity, is a collineation. A sequence of such perspectivities is called a projectivity, and the projectivities may be seen to be exactly those collineations induced by the linear bijections of V. Moreover, the perspectivities may be characterized as those projectivities leaving $P(V) \cap P'(V)$ point-wise fixed. A good geometric discussion of these facts may be found in [S], and a somewhat more algebraic treatment in [A] and [B]. McDonald does not pursue a corresponding analysis of perspectivities over a local ring, presumably because he mainly regards the FT as an algebraic tool for later use in his study of the automorphisms of GL_n.

Although the hypothesis of the FT itself assumes dim $V \geqslant 3$, the above remarks on perspectivities hold also for dim $V = 2$, i.e. in case $P(V)$ is a projective line. In fact, what is known classically as the fundamental theorem of projective geometry is the assertion that there exists one and only one projectivity sending three given distinct collinear points of the projective plane into three other given distinct collinear points. This is an easy consequence of the dim $V = 2$ case of the above theorem that the projectivities are induced by the linear bijections of V.

An interesting feature that emerges when one moves away from a field to a more general coefficient ring is the role played by unimodular elements. The rank 1 free summands of V, i.e. the elements of $P(V)$, are in 1-1 correspondence with the unimodular elements of V, provided one identifies unimodulars that differ by units. Thus, FT may be regarded as a theorem on maps of unimodular elements. (One wonders then about possible generalizations to nonfree modules.)

There are other things related to FT that could be mentioned: for example, some recent versions for dim $V = 2$ over a local ring **[L]**, **[LL]**. However, lest the reader get the impression that the fundamental theorem is the book's main concern, let me hasten to add that the remaining four-fifths of it is devoted to a systematic classification of the normal subgroups and automorphisms of the general linear group, the symplectic group, and the orthogonal group, each group being accorded a separate chapter. I would personally have liked a broader treatment of some of this initial material, though, even possibly at the expense of one of these later chapters.

The book itself is very readable, and McDonald has taken pains to repeat definitions and terminology at about the right intervals to keep them fresh in the reader's mind. The choice of a local coefficient ring is probably a reasonable compromise between the classical theory over a field and the more complicated and less developed theory over, say, a ring of algebraic integers. As presented here, the subject is accessible to first or second year graduate students and would provide an excellent proving ground for the things students at this level may have just learned about groups, rings, modules, and matrices.

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