aim of the subject is to determine values of characters in a block by using this connection. Each p-block B has associated to it a defect group D which is a subgroup of G of order a power of p and determined up to conjugacy. The remarkable results achieved in the case that D is cyclic constitute the high point of the theory and the motivation for much present research. Even more remarkable is the simple combinatorial idea which ties together all these results and all the characters, Brauer characters, decomposition numbers, Cartan invariants and modules in a very simple way: this is the Brauer tree which is a tree together with a planar embedding.

This cyclic theory is the subject matter of the final chapter of the book. Wisely, in this introductory treatment, the authors restrict themselves to the case where a Sylow p-subgroup P of G is of order p; thus, each p-block has defect group of order one or p. Unfortunately, the Brauer tree is not introduced and the reader will not get a complete understanding of the theory.

However, the results on characters are completely established. Recall that the character table of G is a matrix whose rows are indexed by the irreducible characters of G and whose columns are indexed by the conjugacy classes of G. The entry in the row of the character χ and column of the conjugacy class K is the value $\chi(k)$ of χ on an element k of K. In our case, suppose that the characters of degree not divisible by p are listed first and followed by all the characters of degree divisible by p. Similarly, list first the conjugacy classes of elements of order not divisible by p and then the ones of order divisible by p. In this way we get a partition of the character table of G into four submatrices. The main results are then as follows: the lower right submatrix is zero; the upper right submatrix, apart from some signs, is the same as the upper right submatrix of the character table of the subgroup $N(P)$, the normalizer of the Sylow p-subgroup P. This is a beautiful result, easy to understand and very useful in applications; but a whole theory is needed for its proof!

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Applied nonstandard analysis, by Martin Davis, Wiley, New York, London, Sydney, Toronto, 1977, xii + 181 pp., $16.95.

Introduction to the theory of infinitesimals, by K. D. Stroyan and W. A. J. Luxemburg, Academic Press, New York, San Francisco, London, 1976, xiii + 326 pp., $24.50.

Foundations of infinitesimal calculus, by H. Jerome Keisler, Prindle, Weber & Schmidt, Boston, 1976, ix + 214 pp.

Infinitesimal calculus used to be about infinitesimal numbers. A derivative was the quotient of two infinitesimals; an integral was the sum of infinitely many infinitesimals. Although discredited by the development of $\varepsilon - \delta$ analysis in the nineteenth century, the notion of infinitesimals has never entirely disappeared. Physicists continue to draw little vectors and label them

$d\vec{x}$ (just as they used the Dirac delta function long before distributions were invented). I suspect that many mathematicians harbor, somewhere in the back of their minds, the formula $\int\sqrt{(dx)^2 + (dy)^2}$ for arc length (and quickly factor out dx before writing it down). And many students bring to their freshman calculus courses a preconception of the continuum that seems to include infinitesimals; several have told me that the sum of the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ should be just below 2.

Despite its intuitive appeal and its successes in various applications, the calculus of infinitesimals had serious foundational problems. For example, in the calculation of the derivative of x^2,

$$\frac{d(x^2)}{dx} = \frac{(x + dx)^2 - x^2}{dx} = 2x + dx = 2x,$$

dx is treated as zero at the end, but it cannot be treated as zero at the beginning. One thus needs nonzero infinitesimals, numbers which remain negligibly small even when multiplied by large real numbers. No such infinitesimals exist in the real line **R**, so one needs a proper extension of **R**, a rather radical idea, since **R** is almost universally accepted as *the* mathematical model of the continuum. Furthermore, one needs to be able to manipulate infinitesimals and other elements of the extension just like ordinary real numbers, but one cannot expect the extension to satisfy all the usual axioms for **R** (e.g., the Archimedean axiom), for that would make the extension equal to **R**.

In the nineteenth century, it was discovered that there is no need to attack these difficulties. The $\varepsilon - \delta$ method provided definitions, purely in terms of real numbers, of concepts like the derivative and the integral and provided proofs of the classical theorems about these concepts. At the cost of an increase in complexity and perhaps a decrease in intuitive appeal, a rigorous foundation had been constructed for calculus (or, rather, for the part of calculus that did not refer to infinitesimals directly but only via derivatives, integrals, etc.). Infinitesimals were dismissed as fictions; anything one might want to say about them should either be replaced with an ε-δ circumlocution or be judged meaningless. And the problem of creating a usable mathematical theory of the continuum that includes infinitesimals was put on the back burner (or into the freezer) while mathematical logicians (unknowingly) prepared the tools that would eventually lead to its solution.

The development of a rigorous theory of infinitesimals requires a decision as to which properties of **R** shall remain true of the extension, usually called ***R**. The class of such properties must be broad enough to permit computing with infinitesimals as though they were real, but not so broad that they force ***R** to be just **R**. Abraham Robinson [13] discovered that the class of properties expressible by first-order sentences fulfills these requirements beautifully; this was the beginning of nonstandard analysis.

A first-order sentence about a mathematical structure (like **R**) is a sentence built using variables (intended to range over the elements of the structure), names for the elements of the structure, symbols for n-place predicates (i.e., sets of n-tuples, for arbitrary finite n) and functions, propositional connectives (like "and" and "not"), and the quantifiers "for all" and "for some". The crucial point is that, since the variables range only over the structure at hand, a first-order sentence over **R** can express "for all real numbers . . . " but not "for all sets of real numbers . . . ". Thus, the axioms asserting that **R** is an ordered field are first-order, but the least upper bound axiom is not. The compactness theorem of model theory implies that every infinite structure has a proper extension that retains all first-order properties of the original structure. Applying this to **R**, Robinson obtained ***R**, the nonstandard real line. By transfer of first-order properties, ***R** is an ordered field, but it is not Dedekind-complete (a non-first-order property). All predicates and functions on **R** extend to ***R** and retain the same first-order properties, although a predicate will, in general, hold of some elements of ***R** − **R** in addition to those of which it holds in **R**. For example, the Archimedean property of **R**, $\forall x \exists y (N(y) \text{ and } x < y)$ where N is the predicate of being a natural number, continues to hold in ***R**, even though ***R**, being a proper extension of **R**, is not Archimedean. There is no contradiction here, only a proof that the predicate N holds in ***R** of some infinite numbers in addition to the ordinary natural numbers. That is, there are infinitely large natural numbers in ***R**. The natural numbers of ***R** are not well-ordered, for there is no smallest infinite one, but this should not be surprising since well-ordering is not a first-order property.

Robinson also noticed some interesting phenomena that occur when one includes $\mathcal{P}(\mathbf{R})$, the set of all subsets of **R**, in the structure to be extended. One obtains an extension $^*\mathbf{R} \cup {}^*\mathcal{P}(\mathbf{R})$ in which the least upper bound axiom holds, for this axiom is first-order in the base structure $\mathbf{R} \cup \mathcal{P}(\mathbf{R})$. However, this only means that every nonempty bounded element of $^*\mathcal{P}(\mathbf{R})$ has a supremum, where $^*\mathcal{P}(\mathbf{R})$ need not be the set $\mathcal{P}(^*\mathbf{R})$ of *all* subsets of ***R**. Thus, the meaning of the completeness axiom in the extension is weaker than one might at first expect, weak enough to permit ***R** to be a proper extension of **R**. Subsets of ***R** that are in $^*\mathcal{P}(\mathbf{R})$ are called *internal*; the rest are *external*. When sentences are transferred from the standard to the nonstandard universe, quantifiers ranging over sets become quantifiers ranging over internal sets. Thus, nonempty bounded internal sets have suprema, and nonempty internal sets of natural numbers (in ***R**) have least elements. (We have tacitly assumed that $^*\mathcal{P}(\mathbf{R})$ consists of some subsets of ***R**; this can be arranged by replacing $^*\mathbf{R} \cup {}^*\mathcal{P}(\mathbf{R})$ with an isomorphic copy. But note that this replacement will leave us with an embedding, rather than an inclusion, of $\mathcal{P}(\mathbf{R})$ into $^*\mathcal{P}(\mathbf{R})$. This embedding is usually written *. For example, if **Z** is the set of integers, ***Z** is the set of integers of ***R**, including infinite integers.)

One can extend the procedure in the previous paragraph by iterating the power set operation. Davis and Stroyan-Luxemburg (and Keisler in an optional chapter) begin with the "superstructure"

$$\mathbf{R} \cup \mathcal{P}(\mathbf{R}) \cup \mathcal{P}(\mathbf{R} \cup \mathcal{P}(\mathbf{R})) \cup \ldots$$

over **R** (or over some arbitrary set) and form a nonstandard extension of it. (A very similar approach was used by Robinson **[14]**.) It is then possible to give a nonstandard treatment of analysis beyond elementary calculus. For example, the Hilbert space $L^2[0, 1]$ is an element of the superstructure over **R**, and one can consider its nonstandard elements. However, as is shown in [**13**] and, in great detail, in Keisler's book (and the textbook [**6**]), the development of elementary calculus can be carried out comfortably using only ***R**.

The basic concepts of calculus can be defined using ***R** along lines very close to (but not quite identical with) the old pre-ε-δ definitions. For example the derivative dy/dx of a real function is not simply the quotient of infinitesimal differences but rather its standard part, i.e., the real number that is infinitely close to it. (The difference quotient itself is generally in $^*\mathbf{R} - \mathbf{R}$, e.g., $2x + dx$ when $y = x^2$.) If no such real number exists (because the difference quotient is infinite) or if it depends on the choice of the infinitesimal dx, then the derivative does not exist. (Robinson showed that this definition agrees with the conventional ε-δ definition.) Similarly, a definite integral is the standard part of a certain infinite sum of infinitesimals, the number of summands being a nonstandard natural number.

Not only does nonstandard analysis provide a rigorous treatment of infinitesimals in the area of mathematics where they were originally used, it also gives elegant approaches to some ideas that developed later. For example, if X is a topological space, there is a natural way to define what it means for a point of $^*X$ to be infinitely near a point of X; if X is a metric (or merely uniform) space, one can define "infinitely near" even when both points are in $^*X - X$. Then a topological space is Hausdorff if and only if no point of $^*X$ is infinitely near two distinct points of X. It is compact if and only if every point of $^*X$ is infinitely near some point of X. A map $f\colon X \to Y$ is continuous at $x \in X$ if and only if for all $y \in {}^*X$ infinitely near x, $^*f(y)$ is infinitely near $f(x)$. If X and Y are metric spaces, then f is uniformly continuous if and only if for all points $x, y \in {}^*X$ that are infinitely near each other, $^*f(x)$ is infinitely near $^*f(y)$. Once such equivalences are proved (or accepted as definitions), there are trivial proofs of results like the uniform continuity of pointwise continuous functions on a compact metric space. Nonstandard equivalents are known for a great variety of standard concepts, especially in analysis and topology; many of them are given in the books under review, particularly Stroyan-Luxemburg. (It must be admitted that those quoted above were chosen for their elegance. Some of the others are less pleasant.)

Often, as in the examples above, the nonstandard definition of a concept is simpler than the standard definition (both intuitively simpler and simpler in a technical sense, such as quantifiers over lower types or fewer alternations of quantifiers). As a result, nonstandard analysis sometimes makes it easier to find proofs. Also, by providing infinite objects, the nonstandard natural numbers, which behave like finite ones (by the transfer principle), nonstandard analysis sometimes allows one to apply finite results to an infinite situation. A classic example is the Bernstein-Robinson theorem [**1**] asserting that polynomially compact operators on Hilbert space have nontrivial invariant subspaces. Although standard proofs of this theorem and stronger ones now exist, the original proof used nonstandard analysis to carry through the natural idea of applying the well-known existence of invariant subspaces in the finite-dimensional case. One applies it in the nonstandard universe to a Hilbert space whose dimension is a nonstandard natural number and which is carefully chosen to approximate the original (infinite-dimensional standard) Hilbert space.

The relative simplicity of the nonstandard definitions of some concepts of elementary analysis suggests a pedagogical application in freshman calculus. One could make use of the students' intuitive ideas about infinitesimals (which are usually very vague, but so are their ideas about real numbers) to develop calculus on a nonstandard basis. The ε-δ ideas would still be presented, both because they are important in problems of approximation and because students will need them in subsequent courses based on standard analysis, but these ideas, which students often find difficult, would not be used as a foundation on which everything else depends. This approach to freshman calculus is presented in **[6]**, for which Keisler's book under review serves as a supplement for the instructor. Keisler has neatly circumvented the obvious difficulty–that the notion of first-order sentence, which is crucial in the usual development of nonstandard analysis, is inappropriate for freshman calculus. He observed that the uses of the transfer principle that are actually needed at this elementary level are all obtainable from the special case that he calls the solution axiom: if two finite systems of equations and inequalities (involving real constants and functions) have the same solutions in **R**, then they have the same solutions in ***R**. (Actually, as Keisler shows in the book under review, any instance of the transfer principle can be obtained by a sufficiently long string of applications of the solution axiom. But such strings are not needed in **[6]**.) Since students are already familiar with systems of equations and inequalities, and since they are naturally inclined to calculate with infinitesimals exactly as they do with real numbers, the solution axiom seems quite appropriate. (It is, of course, easy to confuse students by concentrating too much on axioms, but this problem arises in standard as well as nonstandard calculus. In fact, the completeness axiom is easier to formulate and apply in the nonstandard form: every finite (i.e., lying between two reals) element of ***R** is infinitely near some real number.)

Calculus courses based on **[6]** require more preparation from the instructor than conventional calculus courses; he must know nonstandard analysis. The syllabus of such a course is also subject to a constraint not found in conventional courses, the need to make contact with the standard approach soon enough for students who will be in a standard course next semester (or quarter). Nevertheless, on the basis of the (admittedly fragmentary) information available to me, it appears that where the nonstandard approach to freshman calculus has been tried it has usually succeeded. Keisler's book provides a valuable service by isolating, and giving a careful elementary exposition of, the part of nonstandard analysis that one ought to know before teaching from **[6]**.

Nonstandard analysis provides natural mathematical models of many situations where one's intuition involves infinite or infinitesimal quantities. For example, such models have been produced in economics, where one thinks of infinitely many individuals each having an infinitesimal impact on the whole economy **[3]**, in probability theory, where one thinks of the probability of an event as the (infinite) number of favorable cases divided by the number of all cases **[2]**, **[7]**, **[9]**, **[12]** and in physics, where one thinks of a quantum field with infinite fluctuations in infinitesimal regions [8]. The growing number of applications of nonstandard methods is likely to convince more mathematicians of the value of learning and teaching these methods; in particular, I expect that nonstandard calculus courses will not always be the rarity they are today.

Of the three books under review, Keisler's is the most elementary, Davis's is next, and Stroyan and Luxemburg's is the most advanced. Keisler deals only with basic calculus, developing it using only a proper extension $^*\mathbf{R}$ of $\mathbf{R}$ satisfying the solution axiom. The possibility of extending a superstructure is discussed only in an optional chapter. The existence of $^*\mathbf{R}$ is established by a simplified version of the usual (Henkin) proof of Gödel's completeness theorem–simplified because of the simple form of the solution axioms. (The optional chapter also gives the ultrapower construction of a nonstandard model.)

Going beyond calculus, Davis has chapters on topological and metric spaces, normed linear spaces, and Hilbert spaces (including the Bernstein-Robinson theorem mentioned above). He also treats many of the calculus topics from a more advanced viewpoint; for example, the Riemann integral is defined for functions into a Banach space. The Stroyan-Luxemburg book contains discussions of a very large number of advanced topics in analysis, especially complex and functional analysis. For these purposes, the nonstandard model is taken to be an extension not just of $\mathbf{R}$ but of the whole superstructure over $\mathbf{R}$ (or over the space being considered). Furthermore, it is not enough to know that $^*\mathbf{R}$ properly extends $\mathbf{R}$; one needs to know that the nonstandard universe is rich in the sense that certain infinite systems of formulas are guaranteed to have solutions provided each finite subsystem has a solution. By specifying the meaining of "certain systems" in various ways, one obtains the definitions of an *enlargement* of a superstructure and of various sorts of *saturated* nonstandard models. Following Robinson [14], Davis uses an enlargement as his nonstandard universe. Stroyan and Luxemburg require only a proper extension of the superstructure in the first half of their book, but in the second half (where the general topology and other abstract topics are) they require a strong type of saturation (which implies enlargement). These models are obtained as ultrapowers (in Davis and the first half of Stroyan-Luxemburg) and ultralimits (in the second half of Stroyan-Luxemburg) of the standard superstructure. Thus, the easiest construction of a nonstandard model is in Chapter 1 of Keisler, the second easiest in Chapter 1* of Keisler and Chapter 3 of Stroyan-Luxemburg, the third easiest in Chapter 1 of Davis, and the hardest (and most powerful) in Chapter 7 of Stroyan-Luxemburg.

Readers who want to learn nonstandard analysis from scratch are advised to begin with Davis or Keisler to learn the basics. Afterward, they can consult Stroyan-Luxemburg (especially the second half) for nonstandard ways of looking at many topics in analysis, or they can turn to the research literature, for example [5], [10], [11].

My reason for not recommending Stroyan-Luxemburg as an introduction to the subject is that I found it very difficult to read. Much of it resembles a rough draft rather than a finished book. A multitude of errors in grammar (especially incomplete and run-together sentences) and usage distract the reader, and in at least one passage the punctuation is so bad that the meaning is altered. There are a number of annoying little slips, like "a real number in $[0, 1] \times \mathbf{R}$," or the use of $\equiv$ in two different senses on two consecutive pages with no explanation in either case. Concepts are often used (even in theorems) before they are defined, or a definition is included as a sort of afterthought in the statement of a theorem. Unusual notation and terminology add to the reader's difficulties. The arrangement of theorems and lemmas seems haphazard in several places. Finally, there are some actual mathematical errors, most of which are attributable to carelessness. For example, the truth definition (on pp. 35 and 36) by induction on sentences (rather than formulas) works only if every element of the model has a name in the language; the last exercise on p. 61 is incorrect if at least two of the three points are infinitely close; $SL(2, \mathbf{C})$ is a double cover of the group of Möbius transformations, not an isomorphic copy (p. 165); and the balls described at the top of p. 203 do not cover the unit ball as claimed. One error, however, seems not to be just careless, and is also committed in passing by Davis (p. 82). Davis implies, and Stroyan and Luxemburg explicitly assert (p. 42) that Łoś's theorem (the fundamental transfer theorem for ultrapowers) can be proved using only the Boolean prime ideal theorem rather than the full axiom of choice. A result of Howard [**4**] shows that this is not correct. One can get nonstandard models using only the Boolean prime ideal theorem, but one needs a Henkin construction rather than an ultrapower. (Incidentally, the proof of Łoś's theorem in Stroyan-Luxemburg is longer and more complicated than the usual proof. The extra complexity may have contributed to the error regarding the axiom of choice.) Finally, I would like to complain about the fairly common but quite illogical convention that constants are zero-place relation symbols (Stroyan-Luxemburg, p. 189); if they are to be viewed as special cases of something else, constants should be zero-place function symbols. Despite all these negative comments, I must recommend Part II of Stroyan-Luxemburg to anyone who has already learned nonstandard methods and some advanced topics in standard analysis and wants to see how they interact; there is a lot of interesting mathematics here.

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