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DETERMINATION OF THE AUGMENTATION TERMINAL FOR FINITE ABELIAN GROUPS

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Let G be a finite abelian group, and let IG denote the augmentation ideal in the integral group ring $\mathbf{Z}G$. The graded ring associated with the filtration on $\mathbf{Z}G$ determined by the powers of IG is

$$\operatorname{gr} \mathbf{Z}G = \sum_{n \geqslant 0} \oplus IG^n/IG^{n+1}.$$

We write $Q_nG = IG^n/IG^{n+1}$. As is well known [1], [6], the sequence Q_nG becomes stationary after a finite number of steps. We call its terminal value the *augmentation terminal*, $Q_\infty G$. We outline here a method for investigating $Q_\infty G$ for any G.

An obvious splitting allows us to assume that G is a p-group.

We choose a generator for each cyclic direct factor of G. Let Γ be our set of such generators, and let $\Lambda = \{\lambda | \lambda + 1 \in \Gamma\}$. Generalizing Lemma 2 of [3] we have

LEMMA. *For $n \geqslant 1$ the set of n-fold products of elements of Λ generates IG^n; a fortiori it generates Q_nG.*

If $\lambda \in \Lambda$ *there is an integer r such that* $(\lambda + 1)^{p^r} - 1 = 0$. Furthermore, by the structure of G, these equations are the only possible source of relations among the elements of Λ. Hence we have immediately

THEOREM 1. *Let $f(\lambda_1, \ldots, \lambda_k)$ be a nontrivial relator in Q_nG, where the $\lambda_i \in \Lambda$. Let $\lambda_i + 1$ be of order p^{r_i} in G, each i. Let $X_1, \ldots, X_k$ be indeterminates over $\mathbf{Z}$. Then there are polynomials $h_i(X_1, \ldots, X_k)$ with integer coefficients such that*

$$f(X_1, \ldots, X_k) - \sum_{i=1} \{(X_i + 1)^{p^{r_i}} - 1\} h_i(X_1, \ldots, X_k)$$

has no terms of degree $\leqslant n + 1$.

Actually using this result to find relators is far from easy, as the references show [2], [7], [8]. If G is an elementary p-group we have

THEOREM 2. *The relators in $Q_\infty G$ are generated by $\{p\lambda | \lambda \in \Lambda\}$ and $\{\lambda^p\mu - \lambda\mu^p | \lambda, \mu \in \Lambda\}$.*

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PROOF. Easy hand calculation shows that these are indeed relators. It is trivial to work out the group given by these relators and then use Theorem 5 of [6] (see also Theorem 3 below).

By much more tedious calculation one can show

LEMMA. *If* $\lambda, \mu \in \lambda$ *and* $\lambda + 1, \mu + 1$ *have order* p^2 *in* G, *then* $\lambda^{p^2}\mu^p - \lambda^p\mu^{p^2}$ *is a relator.*

The appropriate generalization is readily conjectured, but a direct proof is likely to be very difficult.

We developed in [3], [7] and [8] a technique of "standard forms" which is generally suitable for determining the structure of IG^n modulo a given set of relators of Q_nG. This technique may be applied to any G. If the order of the group so determined is the same as that of $Q_\infty G$, then $Q_\infty G$ has been found. Otherwise, another relator must be hunted down.

The order of $Q_\infty G$ is calculated via the module index $[\mathcal{B} \cap \mathbf{Q} . IG: \mathcal{B} . IG]$ where $\mathcal{B}$ is the maximal order in $\mathbf{Q}G$. This follows from [6], where we used our results on invertible powers of ideals [4], [5]. Specifically, by direct calculation from Theorem 5 of [6].

THEOREM 3. *Let* G *be the direct product of* a_i *cyclic groups of order* p^i, $1 \leq i \leq m$. *Then the order of* $Q_\infty G$ *is* p^J, *where*

$$p^J = p^{t_1} + \cdots + p^{t_{m-1}} + \frac{p^{t_m+1} - 1}{p - 1}$$

in which

$$t_i = a_1 + 2a_2 + \cdots + (i-1)a_{i-1} + i(a_i + \cdots + a_m - 1), \qquad 1 \leq i \leq m.$$

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