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THE FUNCTIONS OPERATING ON HOMOGENEOUS BANACH ALGEBRAS[1]

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In this note, we announce a negative solution to the "dichotomy problem" in the context of homogeneous Banach algebras. We begin with the following notation. Let $A(\mathbf{T})$ denote the algebra of absolutely convergent Fourier series, and let $C(\mathbf{T})$ be the class of all continuous functions on $\mathbf{T}$. Let B be a semi-simple, self-adjoint Banach algebra with maximal ideal space $\mathbf{T}$. We view B as an algebra of continuous functions on $\mathbf{T}$. B will be called homogeneous provided the following two properties hold:

(1) For every $a \in \mathbf{T}$, the mapping $f(x) \longrightarrow f(x + a)$ is an isometry of B into itself.

(2) For every $f \in B$, we have

$$\lim_{a \to 0} \|f(x + a) - f(x)\|_B = 0.$$

B will be called strongly homogeneous provided we also have

(3) For every integer k, the operator $f(x) \longrightarrow f(kx)$ maps B into itself and is of norm 1.

It is well known that only analytic functions operate on $A(\mathbf{T})$ (see [1] or [4, Chapter 6]). Clearly, all continuous functions operate on $C(\mathbf{T})$. We have the following "intermediate" result:

THEOREM 1. *There exists a strongly homogeneous Banach algebra B satisfying the following two properties*:

(a) $A(\mathbf{T}) \subsetneqq B \subsetneqq C(\mathbf{T})$.

(b) *Nonanalytic functions operate on B.*

The question solved by this result arose naturally in the study of the operational calculus of $A(\mathbf{T})$ (see [1] and Chapter 6 of [4]). For some previous results related to this question, we refer the reader to [2] where the problem is specifically posed, and to [3].

We now indicate our construction of B. Let $\psi \in P$, the class of trigonometric polynomials. An admissible representation for ψ is defined as an expansion of ψ in the form $\psi = \sum_{k=1}^n a_k \psi_k$, where $\psi_k \in P$, and $\|\psi_k\|_\infty \leqslant 1$, $1 \leqslant k \leqslant n$. Define

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$$\|\psi\| = \inf \sum_{k=1}^{n} |a_k| \log(\|\psi_k\|_{A(\mathbf{T})} + e^2),$$

the infimum taken over all admissible representations $\psi = \Sigma_{k=1}^{n} a_k \psi_k$. It is not difficult to verify that $\|\cdot\|$ is an algebra norm on P and $\|\psi\|_\infty \leqslant \|\psi\| \leqslant 3\|\psi\|_{A(\mathbf{T})}$, for all $\psi \in P$. Let B denote the completion of P under the norm $\|\cdot\|$.

Then B is a strongly homogeneous Banach algebra on $\mathbf{T}$ and $A(\mathbf{T}) \subsetneq B \subsetneq C(\mathbf{T})$. In fact, if $\psi \in B$, then $\{\hat{\psi}(n)\}$ is in the Lorentz sequence space $l_{2,1}$. Property (b) of Theorem 1 is an immediate consequence of the following result:

THEOREM 2.

$$\sup_{\substack{\psi \in B \\ \psi \text{ real} \\ \|\psi\| \leqslant 1}} \|e^{ir\psi}\| \leqslant C_1 |r|^{3/2} \exp(C_2 |r|^{1/2}),$$

for $|r| \geqslant 1$. *Here* C_1 *and* C_2 *are absolute constants.*

The above estimate is obtained by decomposing an admissible representation for a real trigonometric polynomial in a suitable manner. This permits us to exploit the "cancellation" properties of the logarithmic function used in the definition of $\|\cdot\|$. Detailed arguments will appear elsewhere (see [5]).

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