HERMITIAN FORMS AND HIGHER ALGEBRAIC K-THEORY

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Introduction. This note promulgates certain invariants for Hermitian forms over a ring with antistructure which are defined utilizing the algebraic K-theory of the ring in a Z_2-equivariant fashion. Low degree cases of these invariants extend the classical rank, discriminant, and Hasse-Witt invariants.

The principal result described here is that these invariants (using all degrees) yield the stable classification of forms over many fields and algebras for which low degree cases alone do not usually suffice (because of signature invariants that may be defined). The manner in which this is accomplished is quite amusing, in that it sheds light on why the reductions of the signatures modulo certain powers of 2 are in fact determined by the low degree invariants. More general results can be obtained, e.g. for higher unitary algebraic K-theory. However, these are suppressed here in favor of an outline of the equivariant algebraic K-theory upon which this work so crucially relies. This will be dealt with at length in $[\mathrm{Gi}_2]$, and arithmetic aspects will be pursued further elsewhere.

1. Hermitian forms. If R is a ring and M_R is the category of right R-modules, then P_R will denote the full, exact subcategory of M_R determined by the finitely generated projective modules, and $K_n R$, $n \in \mathbf{Z}$, will denote the (Gersten-Wagoner) *algebraic K-theory groups* of R (cf. [Ge]).

Recall that an *antistructure* (α, u) on R consists of an antiautomorphism $r \longmapsto r^\alpha$ of R and a unit $u \in R^\times$ such that $uu^\alpha = 1$ and $r^{\alpha^2} = uru^{-1}$, $r \in R$. An *Hermitian form* over (R, α, u), or an (α, u)-*form* over R, is a pair $(\mathbf{P}, \lambda)$ with $\mathbf{P} \in P_R$ and $\lambda \colon \mathbf{P} \times \mathbf{P} \to R$ such that

$$\lambda(x, y + zr) = \lambda(x, y) + \lambda(x, z)r, \quad \lambda(y, x) = \lambda(x, y)^\alpha u \qquad \text{for } x, y, z \in \mathbf{P}, r \in R,$$

and $(\mathrm{Ad}\ \lambda) \colon \mathbf{P} \to D_\alpha \mathbf{P}$ is an isomorphism where $(\mathrm{Ad}\ \lambda)(x)(y) = \lambda(x, y)$, $x, y \in R$. Here, $D_\alpha \colon P_R \to P_R$ is the α-*conjugate duality cofunctor* $D_\alpha? = \mathrm{Hom}_R(?, R)$ where $\mathrm{Hom}_R(\mathbf{P}, R)$ has right R-module structure $(f \circ r)(x) = r^\alpha f(x)$, $x \in \mathbf{P}, r \in R, f \in D_\alpha \mathbf{P}$. There are evident notions of *isometry* and *orthogonal direct sum* of (α, u)-forms over R (cf. [Wa]), resulting in a permutative category $U(R, \alpha, u)$ and (Gersten-Wagoner type) *unitary algebraic K-theory groups* $KU_n(R, \alpha, u)$, $n \in \mathbf{Z}$, of (R, α, u). Of primary interest here is $KU_0(R, \alpha, u)$ which is just the Grothendieck group of $U(R, \alpha, u)$.

2. Equivariant algebraic K-theory. Refined and adapted from [Va], the following is essential to this study.

THEOREM A. *If R is a ring with antistructure (α, u), then there is a Z_2-spectrum $\mathbf{K}_{\alpha,u}R$, functorial in (R, α, u), defining a Z_2-equivariant generalized cohomology theory $K_{\alpha,u}R_{Z_2}^{\ast}(\ )$ such that:*

(1) as ordinary spectrum $\mathbf{K}_{\alpha,u}R$ is an Ω-spectrum for $K_{\ast}R$, e.g.
$K_{\alpha,u}R_{Z_2}^{-n}(Z_2) = K_n R$, $n \in \mathbf{Z}$, *where Z_2 has free Z_2-action;*

(2) the fixed point subspectrum $(\mathbf{K}_{\alpha,u}R)^{Z_2}$ is an Ω-spectrum for $KU_{\ast}(R, \alpha, u)$, e.g. $K_{\alpha,u}R_{Z_2}^{-n}(\mathrm{pt}) = KU_n(R, \alpha, u)$, $n \in \mathbf{Z}$;

(3) the map $\mathrm{pt} \leftarrow Z_2$ of Z_2-spaces induces the forgetful natural transformation $\varphi_{\ast}: KU_{\ast}(R, \alpha, u) \longrightarrow K_{\ast}R$.

Note. Equivariant cohomology theories are as in [Br].

3. Invariants of forms. Let $D^{p-1}K_n(R, \alpha, u) = K_{\alpha,u}R_{Z_2}^{-n}(S_A^{p-1})$ where S_A^{p-1} denotes the $(p-1)$-sphere with antipodal involution, and set $\hat{D}K_n = \varprojlim_p D^{p-1}K_n$. If $\hat{DK}_n$ and KU_n are filtered by

$$F^p \hat{DK}_n = \mathrm{Ker}(\hat{DK}_n \longrightarrow D^{p-1}K_n) \quad \text{and} \quad F^p KU_n = \mathrm{Ker}(KU_n \longrightarrow D^{p-1}K_n),$$

the latter induced by $\mathrm{pt} \leftarrow S_A^{p-1}$, then $\mathrm{pt} \leftarrow S_A^{\infty-1}$ induces imbeddings

$$\varphi_n^{(p)}: KU_n/F^{p+1}KU_n \hookrightarrow \hat{DK}_n/F^{p+1}\hat{DK}_n \subseteq D^p K_n$$

and hence an imbedding $\hat{\varphi}_n: \hat{KU}_n = \varprojlim_p KU_n/F^{p+1}KU_n \hookrightarrow \hat{DK}_n$.

If $G^p = F^p/F^{p+1}$, then $G^p \hat{DK}_n = E_\infty^{p,-n-p}$ where $E_r^{p,-q}$ is a Bredon-Atiyah-Hirzebruch spectral sequence for $\hat{DK}_{\ast}$ with

$$E_2^{p,-q}(R, \alpha, u) = H^p(Z_2; K_q R)$$

and $K_q R$ has Z_2-action coming from D_α. Hence, restriction $\bar{\varphi}_n^{(p)}$ of $\varphi_n^{(p)}$ imbeds $G^p KU_n(R, \alpha, u)$ in the subquotient $E_\infty^{p,-n-p}$ of $H^p(Z_2; K_{n+p}R)$.

THEOREM B. *For $p = 0, 1, 2$, $\bar{\varphi}_0^{(p)}$ is equivalent to the rank, discriminant, total Hasse-Witt invariant, respectively. (Compare $[\mathbf{Gi}_1]$.)*

THEOREM C. *If $F^\infty = \bigcap_p F^p$, then $F^\infty KU_0(E, \alpha, u) = 0$ provided E is* (1) *an archimedean or local field (locally compact) and α is continuous,* (2) *a finite or global field, or* (3) *a finite dimensional semisimple algebra over one of these fields (α continuous if archimedean or local).*

THEOREM D. *If E is a finite dimensional simple $\mathbf{Q}$-algebra with antistructure (α, u), if v is a real place of centre(E) such that (E_v, α_v, u) is of type $O(\mathbf{R})$, $U(\mathbf{C})$, or $Sp(\mathbf{H})$ with associated signature $\sigma_v: KU_0(E, \alpha, u) \longrightarrow \mathbf{Z}$, and if $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ respectively, then* (1) *there exist*

$$\tau_v^{(p)}: D^p K_n(E, \alpha, u) \longrightarrow K_{\mathbf{F}}^{-n}(\mathbf{RP}^p)$$

with $\hat{\tau}_v = \varprojlim \tau_v^{(p)}$ *defined such that, if* $(\mathbf{P}, \lambda) \in \mathcal{U}(E, \alpha, u)$ *with* $\mathrm{rank}_v = p$ *and* $\sigma_v = \sigma$, *then*

$$\hat{\tau}_v \hat{\varphi}_0 [\mathbf{P}, \lambda] = \frac{\rho + \sigma}{2} + \frac{\rho - \sigma}{2} [\gamma_{\mathbf{F}}^1] \in K_{\mathbf{F}}^0(\mathbf{R}P^\infty) = \mathbf{Z} \oplus \hat{\mathbf{Z}}_2$$

where $\gamma_{\mathbf{F}}^1 = \gamma_{\mathbf{R}}^1 \otimes_{\mathbf{R}} \mathbf{F}$, $\gamma_{\mathbf{R}}^1$ *the canonical line bundle, and*

(2) $\mathrm{Coker}(\sigma_v | F^{p+1} KU_0(E, \alpha, u)) = \mathbf{Z}/2^{1 + \beta(\mathbf{F}, p)}$,

where

$$\beta(\mathbf{F}, p) = \begin{cases} \left[\dfrac{p+7}{8}\right] + \left[\dfrac{p+6}{8}\right] + \left[\dfrac{p+1}{4}\right] & \textit{if } \mathbf{F} = \mathbf{R}, \\[4mm] \left[\dfrac{p+1}{2}\right] & \textit{if } \mathbf{F} = \mathbf{C}, \\[4mm] \left[\dfrac{p+1}{4}\right] + \left[\dfrac{p+3}{8}\right] + \left[\dfrac{p+2}{8}\right] & \textit{if } \mathbf{F} = \mathbf{H}. \end{cases}$$

Corresponding results hold over Dedekind domains, e.g.

$$KU_0(\mathbf{Z}, 1, 1)/F^{p+1}KU_0(\mathbf{Z}, 1, 1) = \mathbf{Z} \oplus \mathbf{Z}/2^{\beta(\mathbf{R}, p)}, \quad p \geqslant 0.$$

Ideally, $F^\infty KU_n = 0$ and $\hat{\varphi}_n : \widehat{KU}_n \cong \widehat{DK}_n$ would always hold, and for "arithmetic" rings these appear to be, respectively, true and plausible. For locally compact fields the "continuous algebraic K-theory" analogues work out nicely.

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