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AMENABLE ERGODIC ACTIONS, HYPERFINITE FACTORS, AND POINCARÉ FLOWS[1]

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1. **Introduction.** In this paper we announce the introduction of a new notion of amenability for ergodic group actions and ergodic equivalence relations. Amenable ergodic actions occupy a position in ergodic theory parallel to that of amenable groups in group theory and one can therefore expect this notion to be useful in diverse circumstances. Here we announce applications to hyperfinite factor von Neumann algebras, skew products, Poincaré flows, and Poisson boundaries of random walks. We remark that from Mackey's virtual group viewpoint, we are considering amenable virtual subgroups of locally compact groups.

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Full details and related results will appear elsewhere.

2. **Preliminaries.** A group is amenable if every continuous affine action (i.e., homomorphism into the affine group) on a compact convex set has a fixed point. There is a strong parallel between homomorphisms in group theory and cocycles in ergodic theory [2], [3], [6], [7]. The condition of amenability for ergodic actions which we now spell out is essentially the condition that every cocycle into an affine group has a fixed element in some appropriate sense.

Let G be a locally compact second countable group, S a standard Borel space with a Borel right G-action, and m a probability measure on S quasi-invariant and ergodic under G. Let E be a separable Banach space, $\mathrm{Iso}(E)$ the group of isometric isomorphisms with the strong operator topology, and E_1^* the unit ball in the dual of E with the $\sigma(E^*, E)$ topology. Suppose $c: S \times G \longrightarrow \mathrm{Iso}(E)$ is a (Borel) cocycle, i.e., for all $g, h \in G$, $c(s, gh) = c(s, g)c(sg, h)$ a.e. Let c^* be the induced adjoint cocycle, i.e., $c^*(s, g) = (c(s, g)^{-1})^*: E^* \longrightarrow E^*$. Suppose $s \longrightarrow A(s)$ is a Borel field of compact convex subsets of E_1^*, i.e., $\{(s, x) | x \in A(s)\} \subset S \times E_1^*$ is Borel. We call $A(s)$ c-invariant if for each g, $c^*(s, g)A(sg) = A(s)$ for almost all s.

DEFINITION 2.1. S is called an amenable G-space if for all E, c, and

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c-invariant $A(s)$ as above, there is a Borel function $f: S \longrightarrow E_1^*$ with $f(s) \in A(s)$ a.e., and such that for each g, $c^*(s, g)f(sg) = f(s)$ a.e.

PROPOSITION 2.2. (i) *If G is amenable, then every ergodic G-space is amenable.*

(ii) *If S is an amenable G-space and there is a G-invariant mean on $L^\infty(S)$, then G is an amenable group.*

(iii) *If $S = G/H$ where $H \subset G$ is a closed subgroup, then S is amenable if and only if H is an amenable group.*

If R is a countable standard ergodic equivalence relation on (S, m) [2], [3], then using cocycles on R one can define amenability of R in an entirely analogous manner. If a countable discrete G acts on S, let R_G be the equivalence relation defined by the orbits. We call S weakly amenable if R_G is amenable. This is readily seen to be equivalent to the conditions of Definition 2.1 holding for all cocycles c satisfying the additional assumption that $c(s, G_s) = I$, where G_s is the stability group at s.

3. Skew products and Poincaré flows. Suppose H is a locally compact second countable group and $c: S \times G \longrightarrow H$ is a cocycle. There are then two important constructions associated to the cocycle. There is the skew product action of G on $S \times H$ defined by $(s, h)g = (sg, hc(s, g))$, and the Poincaré flow [3], or the range-closure [6], of c, which is an ergodic H-space, generalizing the construction of a flow built under a function. Typical questions that arise [6] are to determine when the skew product can be ergodic, and to determine for which groups every action appears as a Poincaré flow of a cocycle on a Z-space, where Z is the group of integers. The latter question, of course, asks for a generalization of Ambrose's theorem.

THEOREM 3.1. (i) *If S is an amenable G-space, then the Poincaré flow of c is an amenable H-space.*

(ii) *If S is an amenable G-space and $S \times H$ is an ergodic G-space, then H is amenable.*

We remark that this theorem implies the impossibility of an action of a nonamenable group with finite invariant measure being the Poincaré flow of a cocycle on a Z-space. It also provides many amenable actions of nonamenable groups. In a positive direction, one has the following.

THEOREM 3.2. (i) *If there is a cocycle $c: S \times G \longrightarrow H$ with ergodic skew product, then every H-space X is a Poincaré flow of a cocycle on a G-space Y. If S and X have finite (σ-finite) invariant measures, so does Y.*

(ii) *The following classes of groups admit a cocycle $c: S \times Z \longrightarrow H$ with ergodic skew product, where S has finite invariant measure: compact; connected* *nilpotent Lie; discrete finitely generated nilpotent; any finite product of the above types.*

4. Hyperfinite factors. Suppose now that G is countable and discrete. The classical Murray-von Neumann construction associates to any ergodic G-space S a von Neumann algebra which is a factor if the action is free. Krieger has modified this construction to obtain a factor even if the action is not free [5]. The following is an analogue of a result of A. Connes on the regular representation of discrete groups [1].

THEOREM 4.1. *A Krieger factor is hyperfinite if and only if the associated ergodic equivalence relation is amenable (i.e., the corresponding action is weakly amenable). If the action is free, this is equivalent to S being an amenable G-space.*

5. Poisson boundaries. If μ is an étalée probability measure on a group G, Furstenberg has defined an associated Poisson boundary (B, m) for the random walk defined by μ [4]. Let $p = \int (m \cdot g)\, d\theta$, where θ is a probability measure on G in the class of Haar measure.

THEOREM 5.1. *(B, p) is an amenable ergodic G-space. Hence, if G is transitive on the Poisson boundary of an étalée measure, then the stability groups are amenable.*

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