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# HILBERT'S TWELFTH PROBLEM AND $L$-SERIES

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Let $k$ be a totally real number field of degree $n \geqslant 2$ with conjugate fields $k = k^{(1)}, \ldots, k^{(n)}$. Let $I(\mathfrak{f})$ denote the group of fractional ideals of $k$ generated by those integral ideals relatively prime to a given integral ideal, $\mathfrak{f}$. Let $S(\mathfrak{f})$ denote the subgroup of $I(\mathfrak{f})$ generated by those principal integral ideals $(\alpha)$ with $\alpha \equiv 1 \pmod{\mathfrak{f}}$. The quotient group $H = I(\mathfrak{f})/S(\mathfrak{f})$ is the ray class group (mod $\mathfrak{f}$) of $k$ and corresponds via class field theory to a totally real abelian extension $F$ of $k$.

We define the character of sign $\lambda(\alpha)$ on $k$ by

$$\lambda(\alpha) = \prod_{j=2}^{n} \operatorname{sgn}(\alpha^{(j)}).$$

Let $\mathfrak{C}_0$ denote the subgroup of all $(\alpha)$ in $S(\mathfrak{f})$ such that $\lambda(\alpha) = 1$ and $\mathfrak{T}$ the set of all $(\alpha)$ in $S(\mathfrak{f})$ such that $\lambda(\alpha) = -1$. It can happen that $\mathfrak{C}_0 = \mathfrak{T} = S(\mathfrak{f})$. The condition that this not occur is that for all units $\epsilon$ of $k$ congruent to 1 (mod $\mathfrak{f}$), we must have $\lambda(\epsilon) = 1$. We assume that $\mathfrak{f}$ satisfies this condition, and let $G = I(\mathfrak{f})/\mathfrak{C}_0$. By class field theory, $G$ corresponds to a real abelian extension $K$ of $k$ which is a quadratic extension of $F$.

For any $\mathfrak{C}$ in $G$, let

$$\zeta(s, \mathfrak{C}) = \sum_{\mathfrak{A} \in \mathfrak{C}} N(\mathfrak{A})^{-s}$$

where the sum is over all integral ideals $\mathfrak{A}$ of $\mathfrak{C}$. Let

$$\epsilon(\mathfrak{C}) = \exp[2\zeta'(0, \mathfrak{C})], \qquad \epsilon = \epsilon(\mathfrak{C}_0).$$

CONJECTURE 1. *The numbers $\epsilon(\mathfrak{C})$ are conjugate algebraic integers in $K$. If $\mathfrak{p}$ is a first degree prime ideal in $\mathfrak{C}$ of norm $p$ then the explicit reciprocity law of class field theory is given by*

$$\epsilon^p \equiv \epsilon(\mathfrak{C}) \pmod{\mathfrak{p}}.$$

Our conjecture thus provides an answer to Hilbert's twelfth problem for totally real fields $k$. The purpose of this note is to present the first numerical example of Conjecture 1 with a nonabelian ground field $k$. Conjecture 1 implies that $\epsilon(\mathfrak{C}\mathfrak{T}) = \epsilon(\mathfrak{C})^{-1}$ is a unit, that

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$$\alpha(\mathfrak{C}) = \alpha(\mathfrak{C}\mathfrak{J}) = \epsilon(\mathfrak{C}) + \epsilon(\mathfrak{C})^{-1}$$

is in $F$ and that

$$g(x) = \prod_{\mathfrak{C}(\bmod \mathfrak{C}_0, \mathfrak{J})} (x - \alpha(\mathfrak{C})) = \sum_{j=0}^{|H|} (-1)^j \theta_j x^{|H|-j}$$

has coefficients in $k = k^{(1)}$.

We take $k = k^{(1)} = Q(\beta^{(1)})$ where

$$\beta = \beta^{(1)} = 3.07911886452947847 \cdots$$

is one of the three real roots of

$$x^3 - x^2 - 9x + 8 = 0.$$

The field $k$ has class-number 3, discriminant $2597 = 7^2 \cdot 53$ ($1, \beta, \beta^2$ form an integral basis) and every unit $\epsilon$ of $k$ has $\lambda(\epsilon) = 1$. Thus we may take $\mathfrak{f} = (1)$; $F$ is then the Hilbert class field of $k$ and $K$ is a sixth degree extension of $k$ which is a quadratic extension of $F$. The group $G$ is cyclic of order 6 and is generated by the element $\mathfrak{C}_1$ containing the unique prime ideal $\mathfrak{p}_2$ in $k$ of norm 2. We let $\mathfrak{C}_j = \mathfrak{C}_1^j$, $0 \leqslant j \leqslant 5$. In particular $\mathfrak{J} = \mathfrak{C}_3$. (Indeed $\mathfrak{p}_2^3 = (\beta)$ and $\lambda(\beta) = -1$.)

The following values of $\zeta'(0, \mathfrak{C})$ were found on a computer which worked internally with an accuracy of about 16 decimal places:

$$2\zeta'(0, \mathfrak{C}_0) = 2.6229258798145494 = -2\zeta'(0, \mathfrak{C}_3),$$

$$2\zeta'(0, \mathfrak{C}_2) = -.72668091960461237 = -2\zeta'(0, \mathfrak{C}_5),$$

$$2\zeta'(0, \mathfrak{C}_4) = -.55674277199362199 = -2\zeta'(0, \mathfrak{C}_1).$$

We put $\epsilon_j = \epsilon(\mathfrak{C}_j)$, $\alpha_j = \alpha(\mathfrak{C}_j)$. We then get

$$\begin{aligned} g(x) &= (x - \alpha_0)(x - \alpha_2)(x - \alpha_4) \\ &= x^3 - 18.718329575489666x^2 + 73.354291283859894x \\ &\quad - 81.914383130290574. \end{aligned}$$

The coefficients of $g(x)$ are supposed to be in $k = k^{(1)}$ (in other words, we are getting a particular embedding of $k$ out of Conjecture 1 as well as a particular embedding of $K$ and $F$). Conjecture 1 yields bounds on $\theta_j^{(i)}$ ($i = 2, 3$) and so leads us to the numbers

$$\beta^2 + 3\beta = 18.718329575489740,$$

$$5\beta^2 + 12\beta - 11 = 73.354291283860260,$$

$$6\beta^2 + 13\beta - 15 = 81.914383130291046,$$

which must be $\theta_1$, $\theta_2$ and $\theta_3$ respectively if Conjecture 1 holds.

It may be checked that any root $A$ of

$$x^3 - (\beta^2 + 3\beta)x^2 + (5\beta^2 + 12\beta - 11)x - (6\beta^2 + 13\beta - 15) = 0$$

does indeed generate $F$ and that either root $E$ of $x + x^{-1} = A$ is a unit in $K$ which in fact generates $K$ over $Q$. Lastly, the reciprocity law is as given by Conjecture 1.

Let $\epsilon' = \epsilon_0\epsilon_2\epsilon_4$. Conjecture 1 implies that $\epsilon'$ is the relative norm of $\epsilon$ from $K$ to a quadratic extension $K'$ of $k$. We have shown without assuming Conjecture 1 that $\epsilon'$ generates the unique quadratic extension of $k$ lying in $K$ and that

$$\epsilon' + (\epsilon')^{-1} = \beta + 1.$$

This serves both as a check on the reciprocity part of Conjecture 1 and on the accuracy of the computation of the numbers $\zeta'(0, \mathfrak{C})$.

Some comments about the actual computation may be useful. The function

$$\left(\frac{2597}{(2\pi)^3}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right)^2 [\zeta(s, \mathfrak{C}) - \zeta(s, \mathfrak{C}\mathfrak{T})]$$

is given by a triple integral of a three-dimensional $\theta$-function and we are interested in the value of this integral at $s = 0$. The triple integral splits into two pieces via the inversion formula for $\theta$-functions. At $s = 0$, one of these pieces splits into an infinite sum of single integrals of the form

$$I_1(a) = \int_0^\infty \exp[-a(x + 2x^{-1/2})]\, dx$$

while the other piece splits into an infinite sum of double integrals of the form

$$I_2(a) = \int_0^\infty \int_a^\infty (xt)^{-1/2} \exp[-t(x + 2x^{-1/2})]\, dt\, dx.$$

The interior integral in $I_2$ for a given $x$ was integrated using the continued fraction expansion of the incomplete gamma function as analyzed by R. Terras [2]. The integral over $x$ in $I_2$ was then computed numerically as was the integral for $I_1$. Several hundred integrals of each type were required in the computation. In the procedure finally used, the field $K$ cost \$7. Still, it would be very worthwhile for future computations to have a rapid accurate algorithm for computing $I_1$ and $I_2$ for a wide range of $a$.

More details regarding this example, examples with real quadratic $k$ and analogies with complex quadratic $k$ will be found in [1].

## REFERENCES

1. H. M. Stark, *L-functions at s = 1. III. Totally real fields and Hilbert's twelfth problem*, Advances in Math. 22 (1976), 64–84. IV (to appear).

2. Riho Terras, *On the convergence of the continued fraction for the incomplete gamma function and an algorithm for the Riemann zeta function* (to appear).

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