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BIFURCATION OF PERIODIC ORBITS ON MANIFOLDS, AND HAMILTONIAN SYSTEMS[1]

BY M. BOTTKOL

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We consider a vector field X_0 having a whole submanifold $\Sigma \subset M$ of periodic points, and ask if any periodic orbits persist under small perturbation, i.e. do all vector fields Y sufficiently near X_0 have periodic orbits lying near Σ. Σ is assumed to be compact. Although in the general case there are simple counterexamples (e.g. on $\Sigma = n$ torus) some natural hypotheses on Σ and the flow of X_0 guarantee periodic orbits for Y, which are thought of as bifurcating off the manifold Σ. Our method here is closely analogous to that of Moser [2], [3], and also his method of averaging on manifolds [1].

In the case of Hamiltonian flows, these methods take on added significance, and the classical action integral makes an appearance. Here the results may be viewed as an extension to S^1-actions of results of Weinstein carried out for Z_n-actions [4], [5].

1. The general case. Let X_0 be a vector field on a manifold M and ϕ^t its induced flow. A nondegenerate periodic manifold of X_0 of period τ is a ϕ^t-invariant submanifold of M such that $\phi^\tau(z) = z$ for all $z \in \Sigma$, and such that 1 is an eigenvalue of $d\phi_z^\tau$ of algebraic multiplicity $k = \dim \Sigma$.

We denote the space of vector fields over M by $X(M)$, having the usual C^k norm $\|\cdot\|_k$. We parametrize a neighborhood of the identity in Diff(M) by a neighborhood of $0 \in X(M)$ by taking a metric and setting $u(z) = \exp_z U(z)$, for $U \in X(M)$ small enough. We define an operator $P(u)$: $X(M) \longrightarrow X(M)$ which transports vectors at z to vectors at $u(z)$ by setting, for $W \in T_zM$,

$$P(u)W = \frac{d}{dh}\bigg|_{h=0} \exp_z(U(z) + hW).$$

LEMMA A. *Let X_0 be a C^{l+1} vector field on M^n generating the flow ϕ^t, having a compact nondegenerate periodic manifold Σ of period 1. Suppose Y is a vector field so that $\|Y - X_0\|_{l+1} < \epsilon$ in some neighborhood of Σ. Then for ϵ sufficiently small, there exists a C^l vector field $V \in X(\Sigma)$, a C^l embedding u: $\Sigma \longrightarrow M$ near the inclusion, and ϕ^t-invariant function λ: $\Sigma \longrightarrow R$ so that*

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(i) $P(u)V(z) = du_z X_0 - \lambda(z)Y(u(z))$,

(ii) $[V, X_0] = 0$,

(iii) $\langle V, X_0 \rangle = 0$ *for* $\langle\ ,\ \rangle$ *a* ϕ^t*-invariant metric on* Σ.

Writing $u(z) = \exp_z U(z), z \in \Sigma$, *let* U_Σ = *component of U tangential to* Σ. *Then u and V above are unique if we impose the normalization*

(iv) $\int_0^1 d\phi^{-t} U_\Sigma(\phi^t(z))\,dt = 0$.

In particular, if $V(\zeta) = 0$, then $u(\phi^t(\zeta))$ is a periodic trajectory of Y of period $1/\lambda(\zeta)$. The proof is to solve the linearized equations and then solve the nonlinear system by iteration. The transport operator $P(u)$ is needed not only to give equation (i) sense, but also its independence of the derivatives of u is crucial to avoid loss of derivatives in the iteration.

COROLLARY. *If* X_0 *has a compact nondegenerate periodic manifold* Σ *of period* τ, *suppose that the flow* ϕ^t *of* X_0 *defines a free* S^1 *action on* Σ. *Then if the Euler characteristic* $E(\Sigma/S^1) \neq 0$, *every Y sufficiently close to* X_0 *has at least* 1 *periodic orbit near* Σ *with period close to* τ.

2. The Hamiltonian case. We now consider a manifold P with a symplectic 2-form Ω, and consider Hamiltonian vector fields X_0, X with Hamiltonian functions H_0, H, which are close to one another. We consider a noncritical energy surface $E_0^c = \{z\colon H(z) = c\}$ of H_0 and the corresponding energy surface E^c of H. We suppose that X_0 restricted to E_0^c has a compact nondegenerate periodic manifold $\Sigma \subset E_0^c$. In order to compare the two flows, we take a diffeomorphism $\beta\colon E_0^c \longrightarrow E^c$ between the two energy surfaces so that $\beta^*X = d\beta^{-1}X(\beta(z))$ is a vector field on E_0^c near X_0, and now apply Lemma A to get $u(z)$, $V(z)$ and $\lambda(z)$ satisfying (a) $duX_0 - \lambda\beta^*X = P(u)V$, (b) $[V, X_0] = 0$, (c) $\langle V, X_0 \rangle = 0$.

Assuming that Ω induces an exact 2-form $d\alpha$ in a neighborhood of Σ, set $\gamma_t = \beta \cdot u \cdot \phi^t(\zeta)$ and introduce the X_0-invariant function $S(\zeta) = \int_{\gamma_t(\zeta)} \alpha$ defined on Σ. $S(\zeta)$ is simply the classical "action" of the closed path $\gamma_t(\zeta)$ on E^c. Using (a) and (b), one finds $QV \lrcorner \Omega = dS$, where $Q\colon TM \longrightarrow TM$ is a nonsingular map near the identity. V is "almost" the Hamiltonian vector field of the function S, and $V(\zeta) = 0$ if and only if ζ is a critical point of S.

One can replace the hypothesis that $\Omega = d\alpha$ with the assumption that Σ is "exact" in the sense of Weinstein [5], [6].

In summary, the number of periodic orbits which persist on a given energy surface under Hamiltonian perturbations can be estimated by

$$\#\text{ critical points of } S \geqslant \text{Cat } \Sigma$$

where "Cat" is the Liusternik-Schnirelman category. However, since S is invariant under S^1-action on Σ induced by X_0, the number of critical points of S can be estimated in terms of the category at the quotient space Σ/S^1; see Weinstein [6].

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COURANT INSTITUTE OF MATHEMATICS, NEW YORK UNIVERSITY, COURANT INSTITUTE, NEW YORK, NEW YORK 10012