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REPRESENTING HOMOLOGY CLASSES BY EMBEDDED CIRCLES AND THE EXISTENCE OF CIRCLES INVARIANT UNDER ISOMETRIES

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ABSTRACT. Paper concerns the problem of representing homology classes by embedded circles, and the question of existence of circles invariant under an isometry of a compact surface.

If I: $\mathbf{R}^3 \longrightarrow \mathbf{R}^3$ is an isometry and $M \subset \mathbf{R}^3$ is an embedded compact invariant surface, then we can prove that there is always a circle on M which is invariant under I. This result follows from Theorem 3 and the fact that any isometry of the sphere or torus has an invariant circle.

Let f: $M \longrightarrow M$ denote an orientation preserving diffeomorphism of finite order on a compact oriented surface, and let P: $M \longrightarrow M_f$ be the natural projection to the orbit or quotient space M_f. We will consider two embedded circles to be equivalent if they are isotopic through invariant circles.

THEOREM 1. *Let f: $M \longrightarrow M$ where $M \neq S^2$. Then*

(1) *There exist an infinite number of distinct homology classes represented by an invariant circle iff $M_f \neq S^2$ or $f^2 = \mathrm{id}_M$.*

(2) *If $M_f = S^2$ and $f^2 \neq \mathrm{id}_M$, each invariant circle disconnects M.*

THEOREM 2. *There exists an f: $M \longrightarrow M$ of order 30 on a surface of genus 11 with the following properties.*

(1) *f has no invariant circle.*

(2) *If g: $M \longrightarrow M$ has no invariant circles then g is conjugate to f^r where r is relatively prime to 30.*

THEOREM 3. (1) *If f: $M \longrightarrow M$ has order $p^k q^l$ where p and q are primes, then f has an invariant circle.*

(2) *If the genus of M is less than 11 then every f: $M \longrightarrow M$ has an invariant circle.*

(3) *If f: $M \longrightarrow M$ is induced by an isometry F: $\mathbf{R}^3 \longrightarrow \mathbf{R}^3$ then f has at least 4 invariant circles when the genus of M is greater than 1.*

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CONJECTURE. There exist an infinite number of compact surfaces such that every diffeomorphism of finite order has an invariant circle.

The proofs of the above theorems are based on the classical theory of branched covering spaces applied to the natural projection $P: M \longrightarrow M_f$. We apply the algorithm in [1] for representing primitive homology classes by embedded circles to prove the existence part of Theorem 1.

The algorithm in [1] can also be applied to give a simple geometric proof of the classical result: Every integer $2g \times 2g$ symplectic matrix is the matrix of $f_*: H_1(M, Z) \longrightarrow H_1(M, Z)$ for some diffeomorphism $f: M \longrightarrow M$ on a surface of genus g. This geometric proof also yields the following.

THEOREM 4. *If $\gamma_1, \gamma_2, \ldots, \gamma_k \in H_1(M, Z)$ are rationally independent classes, then $\gamma_1, \gamma_2, \ldots, \gamma_k$ can be represented by pairwise disjoint embedded circles iff in terms of the standard basis for $H_1(M, Z)$, $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the first k columns of a $2g \times 2g$ symplectic matrix.*

The algorithm in [1] can also be applied to generalize a theorem of Papakyriakopoulous. He proved the next theorem for the case of a homotopy 3-sphere. It seems likely that Theorem 5 applied to a homology 3-sphere of genus 2 would produce a counterexample to his group theory conjecture 2 in [4].

DEFINITION. A system $\{X_i, Y_i | i = 1, 2, \ldots, g\}$ of circles on a surface M of genus g is canonical if $M - \bigcup_{i=1}^{g} \{X_i \cup Y_i\}$ is a sphere with g holes.

THEOREM 5. *Suppose M^3 is a homology 3-sphere and T and T' are solid g-holed handlebodies giving rise to a Heegard splitting of M^3. If $N = \partial T = \partial T'$ then there exist two canonical systems $\{X_i, Y_i | i = 1, 2, \ldots, g\}$ and $\{X_i', Y_i' | i = 1, 2, \ldots, g\}$ for N with the following properties.*

(i) *X_i is contractible in T.*
(ii) *X_i' is contractible in T'.*
(iii) *X_i is homologous to Y_i'.*
(iv) *Y_i is homologous to X_i'.*

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