a thorough discussion of integration over the fiber in a bundle, sphere bundles and the Euler class, the Thom isomorphism, Hopf's theorem on vector fields and Lefschetz's coincidence theorem, ...

The second volume, after an exposition of Lie group theory, introduces principal and associated bundles, connections (principal and linear), parallel displacement, covariant derivative, and curvature; and then discusses the concrete-geometric case of the Weil map, from the point of view of principal bundles and also from that of vector bundles with given structure tensors. One finds there the cohomology of the classical groups (the exceptional groups do not appear) and of some homogeneous spaces, the formulae for the characteristic classes (Pontryagin, Euler, Chern), and Chern's proof for the Gauss-Bonnet-Dyck-Allendoerfer-Fenchel-Weil-Chern theorem.

The third volume, after introductory material on spectral sequences and (very welcome) on Koszul complexes, gives a thorough and complete treatment of the algebraic form of the Weil map. Many examples, classical groups and homogeneous spaces, are worked out. A minor quibble: The third volume does not have an index of notations.

There is a large number of interesting problems in the first two volumes, ranging from simple illustrations to rather difficult general theorems, and adding a lot of "general mathematical education". There is a very extensive bibliography. The third volume has a set of interesting notes on the history of the various facts, and on relations with other topics (e.g., the currently active area of characteristic classes of foliations).

The authors have done us a real service in making this fascinating, but rather complex, field accessible and organizing it so clearly and competently.

H. SAMELSON

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Applications of sieve methods to the theory of numbers, by C. Hooley, Cambridge Tracts in Mathematics, no. 70, Cambridge University Press, Cambridge, London, New York, Melbourne, 1976, xiv + 122 pp., \$18.95.

In number theory there are famous conjectures which can easily be explained even to a layman, but which still resist a complete solution. Two of them are as follows.

There exists an infinity of primes p such that $p + 2$ is also a prime (*the twin prime problem*).

Every even integer greater than 3 is a sum of two primes, or equivalently, every integer greater than 5 is a sum of three primes (*Goldbach's problem*).

It is in the attempt to solve such problems that sieve methods have been developed. The first steps were taken by V. Brun around 1920. Since his pioneering work, there has been progress in refining the techniques and improving the results of sieve theory. The power of the elementary methods originally used has been considerably increased by the combination of analytical arguments. Although sieve methods have not yet sufficed to settle either the twin prime problem or Goldbach's problem, they have led to results rather close to them. The most remarkable achievements, so far, concerning these two conjectures are due to I. M. Vinogradov (1937) and Chen Jing-run (1973). Vinogradov proved, in particular, that every sufficiently large odd integer is a sum of three primes. From Chen's theorem it follows that there are infinitely many primes p such that $p + 2$ has at most two prime factors, and that every sufficiently large even integer is a sum of a prime and an integer with at most two prime factors.

The three main techniques available nowadays to derive sieve results in a general way are known as the *method of Brun*, *the method of Selberg* and *the method of the large sieve*. Before giving a brief, and somewhat crude, account of them, we recall the notion of a sieve.

A *sieve* $\mathcal{S} = (\mathfrak{N}, \mathcal{P}, \Omega_p)$ consists of a finite set $\mathfrak{N}$ of positive integers, a finite set $\mathcal{P}$ of primes, and for every p in $\mathcal{P}$ a set Ω_p of residue classes modulo p. The sieve has "holes" at exactly those integers which lie in a residue class of Ω_p for any p in $\mathcal{P}$. The elements of $\mathfrak{N}$ not falling through the "holes", i.e. not lying in any residue class of Ω_p for p in $\mathcal{P}$, constitute a sequence $\mathfrak{N}_0$, *the sifted sequence*. A sieve is called *small* or *large*, according as the sets Ω_p, p in $\mathcal{P}$, contain a "small" or "large" number of residue classes modulo p.

The main problem in sieve theory is to estimate N_0, the cardinality of $\mathfrak{N}_0$, in terms of the given sieve $\mathcal{S} = (\mathfrak{N}, \mathcal{P}, \Omega_p)$ from above and below. We have

$$N_0 = \sum_{n \in \mathfrak{N}} s(n), \tag{1}$$

where s denotes the *sifting function* defined by

$$s(n) = \begin{cases} 1, & \text{for } n \text{ in } \mathfrak{N}_0, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathcal{S}$ is assumed to be such that Ω_p just contains the residue class 0 modulo p for all p in $\mathcal{P}$, one gets a small sieve for which

$$s(n) = \sum_{d|(n,P)} \mu(d), \tag{2}$$

where μ denotes the Moebius function, (n, P) the greatest common divisor of n and P, and $P = \prod_{p \in \mathcal{P}} p$. With $N_d = \sum_{n \in \mathfrak{N}; d|n} 1$, $d \geqslant 1$, it therefore follows from (1) that

$$N_0 = \sum_{d|P} \mu(d) N_d. \tag{3}$$

Although (3) gives an exact formula for N_0, it usually is of no practical use, since too many summands appear on the right-hand side. However, (3) was the starting point for V. Brun. Indeed, he obtained bounds for N_0 by substituting appropriate partial sums of the right-hand side of (2) for $s(n)$ in

(1). In particular, he proved that for $k = 0, 1, 2, \ldots$.

$$\sum_{\substack{d|P\\ \omega(d)\leqslant 2k+1}} \mu(d)N_d \leqslant N_0 \leqslant \sum_{\substack{d|P\\ \omega(d)\leqslant 2k}} \mu(d)N_d,$$

where $\omega(d)$ denotes the number of distinct prime factors of d. This result follows at once from (1) and

$$\sum_{\substack{d|(n,P)\\ \omega(d)\leqslant 2k+1}} \mu(d) \leqslant s(n) \leqslant \sum_{\substack{d|(n,P)\\ \omega(d)\leqslant 2k}} \mu(d). \tag{4}$$

Inequalities (4), as well as later refinements of the method, were obtained mainly by combinatorial arguments.

In the late forties, A. Selberg introduced a new method to estimate N_0. Unlike his predecessors, he considered not only suitable truncations of the sum in (2) but also looked for simple functions s^+, s^- satisfying $s^-(n) \leqslant s(n) \leqslant s^+(n)$ for all n. In the case of the upper bound, he found the class of functions given by

$$s^+(n) = \Bigg(\sum_{\substack{d\leqslant z\\ d|(n,P)}} \lambda_d\Bigg)^2, \qquad \lambda_d \text{ real}, \lambda_1 = 1, z \geqslant 1, \tag{5}$$

most useful. Defining, then, R_d by

$$N_d = N/f(d) + R_d, \tag{6}$$

where N denotes the cardinality of $\mathfrak{N}$ and f a multiplicative function satisfying $1 < f(d) < \infty$, he obtained

$$N_0 \leqslant N \sum_{\substack{d,d'\leqslant z\\ d,d'|P}} \frac{\lambda_d\lambda_{d'}}{f([d,d'])} + O\Bigg(\sum_{\substack{d,d'\leqslant z\\ d,d'|P}} |\lambda_d\lambda_{d'} R_{[d,d']}|\Bigg), \tag{7}$$

denoting by $[d, d']$ the least common multiple of d and d'. In order to choose the λ_d optimally he looked for a minimum of the quadratic form

$$Q(\lambda) = \sum_{\substack{d,d'\leqslant z\\ d,d'|P}} \frac{\lambda_d\lambda_{d'}}{f([d,d'])}$$

in the variables λ_d under the condition $\lambda_1 = 1$. Such a conditional minimum indeed exists and is given by

$$Q_0 = \Bigg(\sum_{\substack{d\leqslant z\\ d|P}} \frac{\mu^2(d)}{f(d)} \prod_{p|d} (1 - 1/f(p))^{-1}\Bigg)^{-1}.$$

It is attained for

$$(8)\qquad \lambda_d = \mu(d)f(d)Q_0 \sum_{\substack{d'\leqslant z\\ d|d'|P}} \frac{\mu^2(d')}{f(d')} \prod_{p|d'} (1 - 1/f(p))^{-1}.$$

As far as the 0-term in (7) is concerned, it often turns out that R_d is a small remainder term in (6) for an obvious function f. In such cases satisfactory bounds can usually be derived for that 0-term, when the λ_d are given by (8).

Selberg's lower bound method is more intricate than the upper bound one, since we do not know a class of functions s^- analogous in simplicity to the class given by (5). Selberg's idea to by-pass this difficulty is to express s as

$$s(n) = 1 - \sum_{p\in\mathcal{P}} \sigma_p(n)$$

with suitable functions σ_p and to use functions s^- of the form

$$s^-(n) = 1 - \sum_{p\in\mathcal{P}} \sigma_p^+(n),$$

where the σ_p^+ are again amenable to the upper bound method. It is therefore understandable that Selberg's upper bounds are sharper than his lower bounds. But, unfortunately, good lower bounds are required to solve the twin prime problem or Goldbach's problem by sieve methods.

Selberg's method is more powerful and, in many respects, simpler than Brun's. These two methods as described here by means of a sieve with $\Omega_p = \{0 \bmod p\}$ for p in $\mathcal{P}$ can be so adjusted as to yield useful results for small sieves.

The first result for a large sieve was obtained by Yu. Linnik in 1941 and runs as follows. Let $\mathcal{S}$ be a sieve with $\mathfrak{N} = \{1, \ldots, n\}$ and $\mathcal{P} = \{p \text{ prime} | p \leqslant \sqrt{N}\}$. Denote by E_τ, $0 < \tau < 1$, the number of primes p in $\mathcal{P}$ for which Ω_p contains more than τp elements. Then Linnik proved that there is an absolute constant $c > 0$ such that $N_0 \leqslant cN/\tau^2 E_\tau$. In 1948 A. Rényi began his investigations on large sieves. His contributions were partly stronger, partly weaker than Linnik's theorem. Starting with an arbitrary sequence of integers $\mathfrak{M}$: $1 \leqslant m_1 < \cdots < m_Z \leqslant N$ and setting

$$Z(p, a) = \sum_{\substack{m\in\mathfrak{M}\\ m=a(\mathrm{mod}\, p)}} 1,$$

one expects of an evenly distributed sequence $\mathfrak{M}$ that $Z(p, a)$ is about Z/p in size most of the time. Rényi considered the *variance*

$$V = \sum_{p\leqslant X} p \sum_{a=1}^{p} (Z(p, a) - Z/p)^2$$

and proved for $X \leqslant (N/12)^{1/3}$ that

$$V \leqslant 2NZ.$$

Finally in 1965, E. Bombieri and K. F. Roth, independently, sharpened

Rényi's inequality by proving a result comparable with

$$V \leqslant (N + X^2)Z. \tag{9}$$

If $\mathfrak{M}$ is taken to be the sifted sequence $\mathfrak{N}_0$, this also implies Linnik's result. Inequality (9) is a particular case of the inequality

$$\sum_{q=1}^{Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |S(a/q)|^2 \leqslant (N + Q^2) \sum_{n=M+1}^{N+M} |a_n|^2 \tag{10}$$

where S denotes an exponential sum

$$S(x) = \sum_{n=M+1}^{M+N} a_n e^{2\pi i n x}$$

with arbitrary complex coefficients a_n. Similar inequalities have also been proved, when, in the definition of S, characters of some groups are taken instead of $e^{2\pi i n x}$. The proof of such inequalities essentially comes down to an estimate of the operator norm for specific linear maps between finite dimensional Banach spaces.

Inequality (10) contains much more than just an estimate for the sifted sequence of a large sieve. It has been used by Bombieri to prove that for any $A > 0$ there is a $B > 0$ such that for $K = x^{1/2} \log^{-B} x$,

$$\sum_{1 \leqslant k \leqslant K} \max_{2 \leqslant y \leqslant x} \max_{\substack{l \\ (l,k)=1}} \left| \pi(y; k, l) - \frac{1}{\varphi(k)} \int_2^y \frac{du}{\log u} \right| = O(x \log^{-A} x), \qquad x \to \infty, \tag{11}$$

where φ denotes Euler's function and $\pi(y; k, l)$ the number of primes less than y lying in the residue class l modulo k. This is a powerful result. It has been found useful in estimates of the 0-term in (7) and has sometimes even served as a good substitute for the generalized Riemann hypothesis for all Dirichlet L-series. Inequality (10) has also proved useful for small sieves. Nevertheless, (10) is referred to as an *inequality of the large sieve type* in view of its first application (9). The method of the large sieve provides proofs for (10) and similar inequalities.

These three sieve methods are related to Vinogradov's and Chen's theorem, mentioned above, in the following way. Although no one of them enters in Vinogradov's original proof explicitly, Vinogradov estimated certain exponential sums by a method which has some resemblance to Brun's. Another proof of his theorem can now be given with the help of the large sieve. For Chen's theorem the methods of Selberg and of the large sieve are used.

In view of the growing interest in sieve methods, it is not surprising that several books on this subject have appeared in recent years. All three main methods are discussed in [5]. However, the large sieve is only treated in its incomplete form as it was before the contributions of Bombieri and Roth.

Applications are indicated. In [4] the methods of Brun and Selberg are treated and mainly applied to problems of the so-called type H and H_N which generalize the classical twin prime problem and Goldbach's problem. The large sieve method is considered in [1], [3], [6] and [7]. As the main applications of the large sieve, Bombieri's theorem (11) is proved, and upper bounds are given for the numbers of zeros of Dirichlet's L-series in certain domains of the critical strip. These results are of great use for further arithmetical questions.

Hooley's book is not so much concerned with general sieve methods as with various applications of sieve techniques to interesting problems in number theory. Hooley starts with a short, illustrative survey of sieves, stressing the peculiarities of those methods which he is going to use later on. In the rest of his book, these methods are used in connection with problems which include, in particular, the following: Chebyshev's problem on the greatest prime factor of $\prod_{n=1}^{N}(1 + n^2)$, Artin's conjecture on primitive roots, and the problem of Hardy and Littlewood on the representation of an integer as a sum of two squares and a prime. Hooley, himself, has made a substantial contribution to each of them:

If p_N denotes the largest prime factor of $\prod_{n=1}^{N}(1 + n^2)$, then a fragmentary proof of $N = o(p_N)$, $N \to \infty$, was found in Chebyshev's manuscripts after his death. Hooley proves that $N^{11/10} < p_N$ for sufficiently large N, which considerably improves earlier results of A. Markov, T. Nagell and P. Erdös.

Under the assumption of the generalized Riemann hypothesis for certain Dedekind zeta-functions, Hooley gives an asymptotic formula for the number of primes p less than x which admit 2 as primitive root modulo p, when x tends to infinity. Such a formula was conjectured by E. Artin in 1927. However, the numerical factor appearing in it was brought into question by the work of D. H. Lehmer. H. Heilbronn then proposed another factor, which now turns out to be in accordance with Hooley's result.

In 1957 Hooley showed, assuming the generalized Riemann hypothesis for Dirichlet's L-series, an asymptotic formula for the number of times an integer can be represented as a sum of two squares and a prime. In 1960 Yu. Linnik gave an unconditional proof of this result by using his dispersion method. In 1965 another unconditional proof became available, since Bombieri's theorem (11) could now be used instead of the Riemann hypothesis in Hooley's work of 1957. This second proof, which is much shorter than Linnik's, is presented in this book.

Hooley also touches the central problem of the limitation of the methods used. It is a common feature of sieve methods that only upper or lower bounds, rather than the expected asymptotic results, are obtained. An important question therefore is whether, and if so, how far, the bounds can be improved by refinements of the method, e.g., by introducing weights, provided certain natural conditions are fulfilled. The foundations for such investigations were laid in [8] by A. Selberg. In a forthcoming paper [2] E. Bombieri settles that question in the case of the bounds for

$$(12) \qquad \sum_{\substack{n \leqslant x \\ n \in \mathcal{P}_k}} a_n g(n), \qquad x \to \infty, k = 1, 2, \ldots,$$

under some general assumptions on $(a_n)_{n=1}^{\infty}$. In (12), g denotes a suitable weighting function, and $\mathcal{P}_k$ the set of square-free integers having exactly k prime factors. Although one of Bombieri's assumptions is usually not easy to verify for given $(a_n)_{n=1}^{\infty}$, there is no doubt that his work is an important contribution to our knowledge of general sieve methods, which is likely to influence their future development.

Hooley begins the chapters of his book with a historical survey on the relevant problem, and ends them with a discussion of other applications of the method or of possible relaxations of the hypothesis used. This practice is helpful to the reader and provides a good orientation of the subject. The book is written with great attention to detail. It affords an insight into the richness of the problems which can successfully be treated with the help of sieve methods. It can be recommended to anybody interested in sieve methods.

BIBLIOGRAPHY

1. E. Bombieri, *Le grand crible dans la théorie analytique des nombres*, Astérisque, No. 18, Soc. Math. de France, 1974. MR **51** #8057.

2. ———, *The weighted sieve* (to appear).

3. H. Davenport, *Multiplicative number theory*, Markham, Chicago, 1967. MR **36** #117.

4. H. Halberstam and H. E. Richert, *Sieve methods*, Academic Press, New York, 1975.

5. H. Halberstam and K. F. Roth, *Sequences*. Vol. I, Clarendon Press, Oxford, 1966. MR **35** #1565.

6. M. N. Huxley, *The distribution of prime numbers*, Oxford Math. Monographs, Clarendon Press, Oxford, 1972.

7. H. L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Math., vol. 227, Springer-Verlag, Berlin and New York, 1971. MR **49** #2616.

8. A. Selberg, *Sieve methods*, Proc. Sympos. Pure Math., vol. 20, Amer. Math. Soc., Providence, R.I., 1971, pp. 311–351. MR **47** #3286.

A. GOOD

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Abstract analytic number theory, by John Knopfmacher, North-Holland Mathematical Library, vol. 12, North-Holland, Amsterdam & Oxford; American Elsevier, New York, 1975, ix + 322 pp., $29.50.

The reader may wonder what the title of Knopfmacher's book signifies. The word "abstract" refers to an axiomatic set-up of the material which is treated here within the framework of *arithmetical semigroups*, the standard example being the positive integers with their multiplicative structure. The word "analytic" refers to the admission of analytic functions and Cauchy's theorem as tools in proving theorems. Finally "number theory" indicates that this work arose from generalizations of theorems on ordinary integers.

The main topics treated in this book are rooted in:

(i) Dirichlet's theorem that there are infinitely many primes in every residue