modern phase. Most of the results he deals with are over twenty years old and many go back to the thirties and before. Nevertheless the continual process of understanding which has gone on over the years means that clear and concise accounts can now be given.

After a brief introduction the basic notions are introduced in the first chapter (manifolds with boundary are given particular attention throughout). Function-spaces of smooth maps are dealt with next, with originality in the proof of the Baire property for the strong topology and in that of the Morse-Sard theorem on the set of regular values. Sheaf theory in disguise is used to formulate a "globalization theorem" relating the local and global, which is applied in showing that C^r-manifolds ($r > 2$) admit C^∞-structure. Similar ideas are used elsewhere with great advantage, particularly in the proof of the transversality theorem which dominates the third chapter. Vector bundles are dealt with succinctly in the fourth chapter and numerical invariants–degree, intersection number, Euler characteristic–in the fifth. The sixth chapter deals with Morse theory and seventh with the basic notions of cobordism. The last two chapters are concerned with isotopy and the classification of surfaces.

On the whole this is a most readable book. The author has taken considerable trouble with the exposition and has improved on previous accounts in many ways. There are some good diagrams and plenty of exercises. Unfortunately the text contains little in the way of worked examples–the practise as distinct from the theory–and because of this I believe that many students may find these exercises discouragingly difficult. And whatever may be said in the preface, the student is expected to know a little homotopy theory and to be acquainted with the Möbius band, the torus, Klein bottle and so forth. Lapses occur here and there; for instance some key definitions, such as *homotopy*, seem to have been forgotten and others, such as *vector field*, are treated very casually. One would prefer to see more illustrations of the various definitions which come in; for example, one feels that some examples of vector bundles should be discussed before plunging into the general theory. Hirsch's point of view is deliberately restricted by his reliance on certain types of argument–he seems to eschew anything algebraic, such as homology theory–and it would be an improvement if the bibliography included some of the other books in this general area which adopt a different stance. But all in all a most welcome addition to the literature.

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The Hopf bifurcation and its applications, by J. E. Marsden and M. McCracken, Springer-Verlag, New York, 1976, xiii + 408 pp., $14.80.

A classic example from mechanics should make clear what bifurcation from equilibrium of periodic solutions is. Consider a rigid circular hoop so constrained that it can rotate freely about the vertical axis through its center. Suppose a small ball rests at the bottom of the hoop and is constrained to move on the inside rim. Set the hoop to rotate with frequency ω (about the vertical axis through its center). For small values of ω, the ball stays at the bottom of the hoop and that equilibrium position is stable. However, when ω

is increased from 0 to the critical value $\omega_c = \sqrt{g/R}$ (g is acceleration due to gravity, R is the radius of the hoop) and beyond, the ball rolls up the hoop to one of two new stable positions where

$$\cos\theta = g/(\omega^2 R).$$

Here θ is the angle that the radius to the ball makes with the negative vertical axis, the hoop's center being the origin. The position at the bottom of the hoop is still a fixed point, but it has become unstable; and, in practice, it is never observed to occur for $\omega > \omega_c$. Mathematically, we say that the original stable equilibrium has become unstable and has bifurcated into two stable fixed points, one on each side of the hoop, that both correspond to a single periodic orbit of the ball in $\mathbf{R}^3$.

Hopf bifurcation refers to the growth of periodic orbits from a fixed equilibrium point of an autonomous differential system as a continuously changing parameter crosses a critical value. This terminology recognizes E. Hopf's contribution through his definitive 1942 paper [**9**] in which he crystalized and developed earlier ideas and extended them to the case of $\mathbf{R}^n$ ($n > 2$). In $\mathbf{R}^2$ these ideas had their origins in the work of A. A. Andronov and A. A. Witt[1] [**2**], H. Poincaré [**15**] (Poincaré actually was interested in periodic solutions near to nonzero periodic solutions), and A. Lindstedt [**13**]. Hopf was modest to an extreme about his 1942 work, which has generated the research monograph under review: "However, I scarcely think that there is anything new in the above theorem." Indeed, Stokes' investigations of water waves in 1847 [**18**] led him to secular terms, and he found a method to eliminate them; in his proof Hopf also had secular terms to eliminate. Yet, despite its roots in earlier work, Hopf's 1942 paper was almost thirty years ahead of its time. For it was only in the last decade that the significance of his work was recognized.

The book under review may be outdated already as to results, for the explosion of applied mathematical literature on Hopf bifurcation is great and continuing; but this monograph is at the fore with respect to fundamental ideas and methods. (It originated from notes of a 1973–74 seminar at Berkeley, and it contains sections and proofs contributed by many mathematicians.) In the book Hopf's theorem is generalized in various ways: to diffeomorphisms, to partial differential equations, and to situations where various of Hopf's original hypotheses are relaxed. It is also improved. For example, the proof by the center manifold theorem (in §3) establishes uniqueness of the family of bifurcating periodic orbits. Hopf's proof does not preclude existence of a sequence of periodic solutions $x_k(t, \mu_k)$ with $\max|x_k(t, \mu_k)| \to 0$ as $\mu_k \to \mu_{\text{crit.}}$, but with periods $T_k \to \infty$ as $\mu_k \to \mu_{\text{crit.}}$. The center manifold theorem says that any point not on the center manifold must eventually leave a sufficiently small neighborhood of equilibrium (at least for a while) or tend to the center manifold as $t \to \infty$. Thus the 2-dimensional center manifold contains all sufficiently small closed orbits, which implies

[1] Aleksandr Adol'fovich Witt's coauthorship of the famous book by Andronov and Khaikin on nonlinear oscillations deserves to be widely known. This coauthorship was reported by Khaikin in his preface to the second (Soviet) edition (1959), where he wrote that Witt's name was omitted from the title page of the first edition in 1937 "by an unfortunate mistake". Witt died in 1937.

uniqueness. Also Hopf's 1942 paper did not provide an effective formula, foı use in applications, for determining the stability of the periodic orbits bifurcating from equilibrium. Marsden and McCracken fill this gap (in §§4 to 4B), but their goal is wider: "To give a reasonably complete, although not exhaustive, discussion of what is commonly referred to as the Hopf bifurcation with applications." Their goal is solidly achieved. There is a wealth of examples presented which adds significantly to the readability and clarity of the text. These examples make the difference between the motivated and unmotivated, between pedestrian mathematics and stimulating, alive mathematics.

The stability formula derived by Marsden and McCracken in §4, which tells if the bifurcating periodic solutions are asymptotically orbitally stable or not, is quite long; and its derivation is computationally inelegant and lengthy. Several derivations of this result and Hopf's theorem have appeared since 1973. The best I know is due to Y. H. Wan (unpublished; see **[8]** for it and more). He has shortened the derivation significantly and has presented the stability formula and related quantities in a more compact notation. He does this with elegance by using Poincaré normal form and complex-variable notation. Following Wan's ideas, we now outline the content of Hopf's theorem in precise mathematical terms and the stability results derived from it.

Consider the autonomous $n \times n$ ($n \geqslant 2$) system

$$(*) \qquad \dot{x} = A(\mu)x + f(x, \mu)$$

with $f(0, \mu) \equiv f_x(0, \mu) \equiv 0$ and such that (1) A has a pair of complex conjugate eigenvalues $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ and $\bar{\lambda}(\mu)$ with $\alpha(0) = 0$, $\omega(0) = \omega_0 > 0$, and $\alpha'(0) \neq 0$ and (2) for all μ in a neighborhood of 0 the remaining eigenvalues of A all have negative real parts. In this situation, under certain smoothness hypotheses, the improved assertion of Hopf's theorem is that there exists a unique family of small amplitude periodic solutions for values of μ near 0 in exactly one of the cases $\mu < 0$, $\mu = 0$, $\mu > 0$. Moreover, there exist functions $\mu(\varepsilon)$, $\tau(\varepsilon)$, and $\beta(\varepsilon)$ defined for all sufficiently small positive ε such that (1) $\mu(0) = \tau(0) = \beta(0) = 0$, (2) for each such ε, $(*)$ has a unique small amplitude periodic solution of period

$$T = 2\pi[1 + \tau(\varepsilon)]/\omega_0,$$

and (3) the nonvanishing characteristic multiplier, with largest real part (among all the characteristic multipliers of the full system), associated with this solution is $\beta(\varepsilon)$. In Wan's notation the 2×2 system, obtained by applying the center manifold theorem to the given $n \times n$ system $(*)$ if $n > 2$, is of the form

$$\dot{z} = \lambda(\mu)z + g(z, \bar{z}),$$

where $z = y_1 + iy_2$ (y_i real). He then puts this system in Poincaré normal form:

$$\dot{\xi} = \lambda(\mu)\xi + c_1(\mu)\xi^2\bar{\xi} + c_2(\mu)\xi^3\bar{\xi}^2 + \mathcal{O}(|\xi|^7),$$

ξ a complex variable. Wan has shown that if A and f are smooth enough

$$\beta(\varepsilon) = \sum_{i=2} \beta_i \varepsilon^i, \quad \mu(\varepsilon) = \sum_{i=2} \mu_i \varepsilon^i, \quad \text{and } \tau(\varepsilon) = \sum_{i=2} \tau_i \varepsilon^i,$$

where

$$\beta_2 = 2 \operatorname{Re} c_1(0), \mu_2 = -\beta_2 / (2\alpha'(0)), \tau_2 = -[\operatorname{Im} c_1(0) + \mu_2 \omega'(0)]/\omega_0.$$

If ε is sufficiently small and $\beta_2 \neq 0$, then sign μ_2 determines the direction of bifurcation and sign β_2 determines the stability of the periodic solutions. They are asymptotically orbitally stable, with asymptotic phase, if $\beta_2 < 0$. B. D. Hassard and Wan **[8]** have shown that $\mu_3 = \beta_3 = \tau_3 = 0$, and that

$$\alpha'(0)\mu_4 = -\operatorname{Re} c_2(0) - 2 \operatorname{Re} c_1'(0)\mu_2 - \alpha''(0)\mu_2^2/2,$$
$$\beta_4 = 4 \operatorname{Re} c_2(0) + 2 \operatorname{Re} c_1'(0)\mu_2,$$

and

$$\omega_0\tau_4 = -\operatorname{Im} c_2(0) + \omega_0\tau_2^2 - \omega'(0)\mu_4 - \operatorname{Im} c_1'(0)\mu_2 - \omega''(0)\mu_2^2/2.$$

Indeed, it is shown in §3B (for μ_k and β_k) and in **[8]** that, given enough differentiability, $\mu_k = \tau_k = \beta_k = 0$ for all odd k.

§3A contains a brief discussion of N. Chafee's theorem **[4]** which tells how (if $\alpha'(0) = 0$) uniqueness of the family of bifurcating closed orbits, whose existence is asserted by Hopf's theorem, is lost. Chafee only assumes that A and f are continuous in μ and f is uniformly Lipschitzian in x with respect to μ, with Lipschitz constant $k(x) \to 0$ as $|x| \to 0$.

Hopf's theorem is reproved in §3C under the assumption that no "other" eigenvalues of $A(0)$ are integral multiples of $i\omega_0$ and also in the exceptional case $\alpha'(0) = 0$ but $\alpha''(0) \neq 0$ (cf. Hassard and Wan **[8]**). Actually, the two families of periodic solutions derived there (in the $\alpha'(0) = 0$, $\alpha''(0) \neq 0$ case) are one and the same set of periodic orbits.

The most interesting example discussed by Marsden and McCracken is that of the Lorenz equations **[14]** which model, in an idealized way, turbulence in the atmosphere. These equations are

$$(**) \qquad \begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy. \end{aligned}$$

Here $\sigma = \nu/K$ is the Prandtl number, K is the coefficient of thermal expansion, ν is the viscosity, and r, the Rayleigh number, is the bifurcation parameter. Lorenz **[14]** writes that " . . . x is proportional to the intensity of the convective motions, while y is proportional to the temperature difference between the ascending and descending currents, similar signs of x and y denoting that warm fluid is rising and cold fluid is descending. The variable z is proportional to the distortion of the velocity profile from linearity, a positive value indicating that the strongest gradients occur near the boundaries."

At first, and even at second, glance there appears to be nothing remarkable about the system (**). Who among us would have imagined that this seemingly simple autonomous nonlinear system could be associated with Cantor sets? Such pathology in the phase portrait of a 3×3 autonomous system with polynomial right-hand sides almost defies imagination. Yet, this is actually the case; numerical studies show that the Lorenz equations provide an example of a new object, a strange attractor, the subject of much current study. Parts of a strange attractor are locally

$$\text{(a Cantor set)} \times \text{(a disc)}.$$

What is also noteworthy is that E. Hopf's goal in 1942 was to give a mathematical theory of turbulence and that Lorenz's equations exhibit Hopf bifurcation.

Marsden and McCracken applied the Hopf theorem and their stability formula to (**). The stability criterion they obtained was programmed on a computer, yielding regions of stability and instability in the (b, σ)-plane. Their picture shows that if bifurcation to unstable periodic solutions occurs for $r < r_{\text{crit.}}$ about a stable equilibrium, then for $r > r_{\text{crit.}}$ the Lorenz attractor (strictly speaking, invariant set) appears about unstable equilibrium.

In the last half of the book the authors take up the theme provided by their earlier study of Lorenz's equations in earnest. They first remark that Hopf's method has been pushed through for partial differential equations, "provided the equations are of a certain 'parabolic' type. This was done by Judovich **[2]**, Iooss **[10]**, and Joseph and Sattinger, **[11]** and others. In particular, the methods do apply to the Navier-Stokes equations. The result is that if the spectral conditions of Hopf's theorem are fulfilled, then indeed a periodic solution will develop, and, moreover, the stability analysis given earlier applies. The crucial hypothesis needed in this method is analyticity of the solution in t. Here we wish to outline a different method for obtaining results of this type . . . instead of utilizing smoothness of the generating vector field, or t-analyticity of the solution, we make use of smoothness of the flow. ..." In the end, this approach of the authors depends on a "remarkable property of smooth semiflows" **[3]**, which is proved in §8A: "this is that the semiflow F_t is generated by a C^∞ vector field on the finite dimensional center manifold C; i.e. the original $X[\dot{x} = X(x, \mu)]$ restricts to a C^∞ vector field (defined at all points) on C. This trick reduces us to the Hopf theorem in two dimensions. ..."

§§8–9B contain the real meat of the applications portion of the book: applications to problems for the Navier-Stokes equations. Many of the results are not new (D. Ruelle and F. Takens in their fundamental paper **[16]** gave a simple proof of W. Velte's results **[19]** on stationary bifurcation in the flow between rotating cylinders from Couette flow to Taylor cells). But new results are presented, including results on existence, smoothness, and uniqueness. About turbulence the authors make the following interesting remarks: "If the attracting set of the flow, in the space of vector fields which is generated by the Navier-Stokes equations is strange, then a solution attracted to this set will clearly behave in a complicated, turbulent manner. While the whole set is stable, individual points in it are not. Thus an attracted orbit is constantly near unstable (nearly periodic) solutions and gets shifted about the attractor in an aimless manner. . . . " In summary the authors' view of turbulence is: "Our solutions for small μ (= Reynolds number in many fluid problems) are stable and as μ increases, these solutions become *unstable* at certain critical values of μ and the solution falls to a more complicated *stable* solution; eventually, after a certain (finite) number of such bifurcations, the solution falls to a strange attractor (in the space of all time dependent solutions to the problem). Such a solution, which is wandering close to a strange attractor, is called *turbulent*. The fall to a strange attractor may occur after a Hopf bifurcation to an oscillatory solution and then to invariant tori, or may appear by some other mechanism, such as in the Lorenz equations as explained above ('snap through turbulence')."

The authors state two conjectures and two major open problems: "1. In the Ruelle-Takens picture, global regularity for all initial data is not an á priori necessity; the basins of the attractors will determine which solutions are regular and will guarantee regularity for turbulent solutions (which is what most people now believe is the case). 2. Global regularity if true in general, will probably never be proved by making estimates on the equations. One needs to examine in much more depth the attracting sets in the infinite dimensional dynamical system of the Navier-Stokes equations and to obtain the á priori estimates this way. (i) Identify a strange attractor in a specific flow of the Navier-Stokes equations (e.g. pipe flow, flow behind a cylinder, etc.). (ii) Link up the ergodic theory on the strange attractor with the statistical theory of turbulence."

In connection with (ii) R. Williams [**20**] has shown that strange attractors come in uncountably many topological types. Thus there is the further difficulty: does one need a statistical mechanics for each type? We note that no one has proved, as of this writing, that any faithful model of a natural system possesses the strange attractor property for some particular open set of numerical coefficients.

There are important contributions to bifurcation theory (not necessarily Hopf bifurcation) that are not described in detail in these notes. Not everything in this large field could be included. One prominent example is the paper of M. Crandall and P. Rabinowitz [**5**]. Another exposition is by D. H. Sattinger [**17**]. There have been numerous successful applications of the Hopf theorem and stability results to biological problems in the last few years.

Important problems remain to be solved. How do bifurcating solutions of particular systems behave in the large as the bifurcation parameter moves away from its critical value, and what is their stability? J. Alexander and J. York [**1**] have given a deep but partial answer: they give the theoretical possibilities but no method for deciding among them in concrete cases. B. D. Hassard [**7**] has numerically investigated this problem for the space-clamp case of the Hodgkin-Huxley equations, which describe propagation of signals along a squid's giant nerve axon. No successful method has been found for establishing the stability of large periodic solutions of natural systems, namely those faithfully modeling natural phenomena. Further study of the Poincaré map may be the key to resolving this question. Much remains to be done in investigating the connection between periodic and close to periodic solutions of partial differential equations and periodic solutions of corresponding systems of ordinary differential equations. A step in this direction has been taken by E. Conway, D. Hoff, and J. Smoller [**6**], who have proved for a wide class of nonlinear reaction-diffusion systems that solutions exist which decay at an exponential rate in t to stable invariant sets of corresponding systems of ordinary differential equations, for example, to stable bifurcating orbits.

This book should have a healthy influence on the course of bifurcation theory. The range of current applications of the theory is exceptionally large: fluid dynamics, atmospheric physics, buckling of materials, mathematical chemistry, biochemistry, biology and neurophysiology, chemical engineering, mathematical models of morphogenesis, population dynamics, and mathematics, which is surely not a complete list. One may expect that the sciences will strongly benefit, perhaps through a key advance, from applications of Hopf bifurcation and its generalizations and relatives.

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