Entire holomorphic mappings in one and several complex variables, by Phillip
   A. Griffiths, Ann. of Math. Studies, no. 85, Princeton Univ. Press, Prince-
   ton, N. J., 1976, x + 99 pp., $11.50 (cloth) and $4.50 (paper).

This little book is based on the fifth set of Hermann Weyl lectures, which
Phillip Griffiths delivered at the Institute for Advanced Study in the Fall of
1974. And although the notes, because of their informal nature, have some
unclear moments they also capture a broad and enthusiastic modern perspec-
tive on a very classical field.

The history of the subject begins with E. Picard's finding in 1879 that if f is
a nonconstant entire function of one complex variable, then the range of f
can omit at most one finite complex number w. This sensational result
attracted much attention from E. Borel, J. Hadamard, G. Valiron and many
others around the turn of the century. However, Rolf Nevanlinna's study, *Zur
Theorie der meromorphen Funktionen* [21], in 1925 completely revolutionized
the subject. In this article, he developed his own so-called first and second
fundamental theorems, which form the basis of all further research in value-
distribution theory. Indeed, as Weyl himself has written (albeit 34 years ago
[30, p. 8]): 'the appearance of this paper has been one of the few great
mathematical events in our century.' And within the next few years, Nevan-
linna's brother Frithiof and student Lars Ahlfors had obtained their own
derivations of the fundamental Nevanlinna theorems. The echoes of these
techniques (especially those of Ahlfors) reverberate clearly in this book.

The elegance and depth of this theory have naturally led to attempts to
obtain analogues for higher dimensions. The general problem is to consider a
nondegenerate holomorphic mapping

$$(1) \qquad\qquad\qquad f: C^n \to M$$

with M a compact complex manifold of dimension m (one can also consider
more general domains than C^n). Success here, of course, has come more
slowly; in addition to the bibliographical notes appended to each of the
chapters in the book under consideration, we refer to [12] for developments
before 1969 and W. Stoll's recent survey article [27].

Instead of asking only how many points of M are covered by f and how
often, situation (1) allows us, in addition, to consider the covering properties
of any collection V_k of k-dimensional analytic subobjects.

$$(2) \qquad\qquad\qquad V_k \subset M \qquad (k \leqslant m).$$

The first substantial body of results without $\min(m, n) = 1$ in (1) is due to
Stoll and his associates (cf. [25], [26]), and, in particular, Stoll has obtained
first and second fundamental theorems which anticipate many of those
presented in Griffiths' book.

Stoll's proofs (especially of his second fundamental theorem [25]) are based
on the so-called associated maps of Ahlfors and Weyl [3], [30]. A decade
later, R. Bott and S. S. Chern introduced a different perspective in these
matters, and obtained a new first fundamental theorem. Their point of view
has had a profound influence on Griffiths.

The recent work of Griffiths, in association with J. Carlson [8] and J. King [19], has given a new and very direct approach to the second fundamental theorem in many important situations. This book presents an account of these successes in the special case of positive line bundles and $m = n$ in (1), so that $k = n - 1 = m - 1$ in (2). (The methods also apply to the line bundles over projective algebraic varieties.) This work has attracted a great deal of interest, and made the subject one of intense activity. In particular, the techniques introduced work with range M much more general than projective space $\mathbf{P}^m$, and also seem relevant in areas of mathematics other than value distribution theory. Since there still remain many open problems (the situation for general n, m and k in (1) and (2) is not yet settled, even when $M = P^n$), the appearance of this book now is most appropriate.

According to Griffiths, the notes have three goals. The first is to record the successes of [8] and [19] in more permanent form. In addition (goal two) there is conscious attempt to show that the algebro-geometric formalisms used in the modern theory are natural analogues of what is standard in the so-called *classical case*; i.e. when $n = 1$ in (1) and $M = P^1 = \hat{C}$. Finally, the pivotal roles that *growth* and *negative curvature* play in these matters are strongly emphasized. Griffiths has already written several shorter survey articles which also touch on these matters (cf. [15], [16], [17]) but this presentation is more complete and definitive.

The book is written in an inviting and pleasant manner, and the author is willing to volunteer many insights. However, an interested reader who is not sophisticated in complex manifold techniques will find parts very difficult, although the density of these difficulties decreases as one reads on. The texts [11] and [29] and the opening chapter of [19] are given as general references for this material, but a few extra lines or references in appropriate places could have been very helpful. For example, one of the key techniques used by Griffiths is the construction of n-dimensional (pseudo)-volume forms and for general M they depend on a potential-theoretic lemma of Kodaira (p. 17). (Note: for $M = P^n$, this basic volume form can be explicitly given.) The proof of the lemma given here takes but one page with no references, and appeals to such concepts as 'standard Kähler identities' with no further citation. The introduction of the dual of the canonical line bundle (in p. 48) is likewise unheralded, although this notion is important in as basic a matter as the proof of the n-dimensional defect relations.

The chapters are *Order of growth*, *The appearance of curvature* and *The defect relations*.

Chapter one centers on the first fundamental theorem and associated material, and on 'E. Borel's proof' of the Picard theorem. The definition of $T_f(L, r)$, where L is a positive line bundle, appears in p. 18; seven pages later we see how this generalizes the Ahlfors-Shimizu form of Nevanlinna's characteristic $T(r)$. The relation between this theory and the classical one is made very striking by a very frequent and systematic use here and later of so-called Crofton-type formulas; these express many of the n-dimensional integrations as the product of a C^1 integration with one over P^{n-1}. In particular the proof of the first fundamental theorem (and, in the next chapter, the second fundamental theorem) follows from the case $n = 1$, and aside from a step on

p. 60 the Lelong theory of currents is not required here. This method also adapts smoothly to obtain some special cases of the Stoll analysis of growth of analytic sets. (The use of one-variable methods in this manner goes back nearly 40 years. It also plays a key role on Ahlfors' famous article [3].) The chapter closes with a slick sketch of the n-dimensional version of Hadamard's factorization theorem (due to Stoll and Lelong); this is one of the high spots of the book.

Chapter two starts with a special case of the one-variable second fundamental theorem. The proof follows from that used for the first fundamental theorem once it is known that the range M in (1) is endowed with a metric of (strictly) negative curvature. With this as motivation, a study of n-dimensional volume and pseudo-volume forms is made, culminating in a general second fundamental theorem at the end of this chapter. The volume forms used are those of [8], [19], and one is grateful for the care shown in relating these to the spirit of Ahlfors' own now-classical approach [1]. A few dividends such as the Schottky-Landau theorem and big Picard theorem are also included, and there is an illuminating discussion of the relation between the method of negative curvature and the Ahlfors extension of Schwarz's lemma.

A reader not conversant with these matters may fail to see the full importance of the second fundamental theorem in its pregnant form in Chapter two. The major applications (defect relations) appear at the outset in *Chapter three*, using the machinery developed in the first two chapters. The main application is when $M = P^n$ and our line bundle is the hyperplane bundle.

Even in the classical case, the defect relations are most easily obtained by using carefully-chosen volume forms (this is one of Ahlfors' contributions in [1]). However, R. Nevanlinna's original proof was based on his celebrated 'lemma of the logarithmic derivative.' An n-variable version of this lemma is obtained here by the method of 'almost negative' curvature; this version improves that of [19]. We are next treated to Nevanlinna's original derivation of the defect relations, based on the lemma of the logarithmic derivative. After a third proof of the defect relations (this one due to Ahlfors), the book closes with the more contemporary 'ellipse theorem' of A. Edrei and W. H. J. Fuchs (the figure on p. 83 conflicts with this title). The exposition follows that in Hayman's text [20], and thus depends on a complicated integral inequality of A. Goldberg (a proof which avoids this inequality appears in Lemma 1 of [13]).

There is no discussion of open problems and, almost as a corollary, little consideration of the nonequidimensional case in (1). But value distribution theory in that setting also has a long history, especially when $n = 1$ in (1) and $M = P^n$ (for contemporary accounts cf. [18], [31]). This study was initiated by Borel and A. Bloch, before Nevanlinna, who obtained results of Picard type. The work of Ahlfors and the Weyls [3], [30] has already been cited, but the ideas of H. Cartan [9] should also be mentioned. These are based directly on the lemma of the logarithmic derivative, and they play a crucial role in A. Vitter's hyperplane defect relations [27] for maps $f\colon C^n \to P^m$ ($m = n$ is covered in this book). There is also no mention of defect relations when the

V_k in (2) are nonanalytic, and have instead real dimension r $(1 \leqslant r \leqslant 2m - 1)$; that such results may exist is suggested by Ahlfors' theory of covering surfaces [23, Chapter 13].

The book has many minor errors, and some may cause confusion. The '$O(1)$' in (1.15) and (1.16) does definitely depend on $D \in |L|$, as the proof of (1.15) shows. The (Borel) proof of Picard's theorem presented here in Chapter one is less subtle than that given by Borel [4], [5], [6], and depends on the first fundamental theorem in an interesting way. There are several misnumbered formulas, references and careless notations. A cluster of these mar the proof of the big Picard theorem in Chapter two.

This book does not attempt to supplant the existing standard texts [14], [20], [22], [23], but is a valuable supplement to these and a full introduction to a rapidly growing area of activity. It has an up-to-date bibliography, and comes at a favorable price.

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Fuzzy sets and their applications to cognitive and decision processes, edited by L. A. Zadeh, K.-S. Fu, K. Tanaka and M. Shimura, Academic Press, New York, 1975, x + 496 pp., $16.00.

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Scientists have long sought ways to use the precision of mathematics to tame the imprecisions of the real world. One may see many-valued logic, topology, and probability theory as different attempts to be precise about imprecision. In 1965, Lotfi Zadeh [1] suggested that the proper tool for handling imprecision was to replace the rigid all-or-none of set membership by *graded* membership–so that the characteristic function $\chi_A: X \to \{0, 1\}$ of a set in the universe X was to be replaced by a membership function $\chi_A: X \to [0, 1]$ with weights falling in the interval $[0, 1]$. Set operations then generalize as follows:

$$\chi_{A \cup B}(x) = \max[\chi_A(x), \chi_B(x)], \qquad \chi_{A \cap B}(x) = \min[\chi_A(x), \chi_B(x)],$$

$$\chi_{\overline{A}}(x) = 1 - \chi_A(x).$$