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OBSTRUCTED ONE DIMENSIONAL LOCAL COMPLETE INTERSECTIONS

BY GAYN B. WINTERS

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The examples of obstructed one dimensional local complete intersections we have are in fact fibers of a proper map from a smooth surface onto a smooth curve over an algebraically closed field k. The cohomological obstructions to deforming such a fiber X infinitesimally can be seen to lie in the group $\mathrm{ob}(X) = H^1(X, T^1_X)$ where T^1_X denotes the first cohomology sheaf of the cotangent complex of X over k (cf. [2]). We call $\mathrm{ob}(X)$ the *obstruction group* of X. One says that X is *unobstructed* if these obstructions always vanish—this is equivalent to the smoothness of the infinitesimal deformation functor Def_X of X. Of course this is the case when the entire obstruction group $\mathrm{ob}(X)$ is zero; but in our special set-up, the converse actually holds:

PROPOSITION 1. *Let X be a fiber of a proper map ϕ from a smooth surface onto a smooth curve. Suppose the general fiber of ϕ is smooth. Then X is unobstructed if and only if the obstruction group* $\mathrm{ob}(X) = 0$.

To compute whether or not $\mathrm{ob}(X)$ is zero for a given fiber X, write X as a Weil divisor $X = \Sigma\, m_iC_i$ on the surface in which X lies. Set $D = \Sigma\, n_iC_i$ where $n_i = m_i$ if the characteristic divides m_i and $n_i = m_i - 1$ in general.

PROPOSITION 2. *Notation as above, there is a noncanonical isomorphism* $\mathrm{ob}(X) \simeq H^1(D, O_D)$.

Fortunately one can simplify this computation by blowing down any exceptional curves which might be in the fiber X to obtain the "minimal model" X' and then applying:

PROPOSITION 3. *Notation as above, there is a smooth map of functors* $\mathrm{Def}_X \longrightarrow \mathrm{Def}_{X'}$. *Thus, X is obstructed if and only if the minimal model X' is obstructed.*

Now suppose g.c.d.$(m_i) = d > 1$ and the genus of X is $p_a(X) = g \geqslant 1$, then if Z is a divisor such that $dZ = X$, $\chi(O_Z) = \chi(O_X)/d = (1-g)/d$, and since $d > 1$, $h^1(O_D) \geqslant h^1(O_Z) \geqslant 1 - (1-g)/d \geqslant 1$, thus we have:

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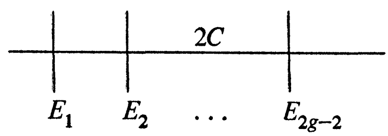
COROLLARY. *X as above, if g.c.d.$(m_i) > 1$ and the genus $p_a(X) \geqslant 1$ then X is obstructed.*

One can use the tables of Kodaira [1] which classify the genus one fibers to see that in characteristic zero the fibers with g.c.d.$(m_i) > 1$ are precisely multiples of a nonsingular elliptic curve, of a nodal rational curve, and of a loop of nonsingular rational curves. In characteristic p, a p^n-fold multiple of the others can also occur. When the g.c.d.$(m_i) = 1$ it is easily checked (either directly or by passing to the minimal model) that the obstruction group vanishes. Thus in all characteristics we have:

COROLLARY. *X as above with genus $p_a(X) = 1$. Then X is obstructed if and only if g.c.d.$(m_i) > 1$.*

In genus two—where the g.c.d. of the multiplicities of any fiber is always one—the lists of Ogg [4] and of Namikawa and Ueno [3] can be used in conjunction with the above two propositions to determine the obstructed fibers of genus two. In the notation of [4], they are the ones whose minimal models have numerical types Nos. 12, 15, 16, 17, 18, 24a, 24, 26, 27, 28, 30, 31, 32, and in characteristic two 19 and 20 also.

One can use the results of [5] to show the existence of obstructed fibers in every genus $g \geqslant 1$. For example, the configuration



where C is elliptic and the E_i are rational represents an obstructed fiber in genus $g \geqslant 1$.

REMARKS. The proofs of Propositions 1, 2, and 3 relate the deformations of the formal completion of the surface along the fiber X to the deformations of X. While direct proofs are probably possible, this method gives additional information which appears to be useful in analyzing the structure of the formal moduli space of X.

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DEPARTMENT OF MATHEMATICS, COLLEGE OF THE HOLY CROSS, WORCESTER, MASSACHUSETTS 01610