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EVERY FINITE LATTICE IS A FACE LATTICE

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ABSTRACT. Given a finite lattice L, there is a finite-dimensional convex set C such that L is the lattice of faces of C.

If C is a convex set, we denote by $L(C)$ the lattice of convex subsets of C. The lattice of (algebraic) faces of C is available from $L(C)$ as $\{x \in L(C): (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x)$ for all a, b in $L(C)\}$. We shall call this face lattice $DL(C)$. It is a complete meet-sublattice of $L(C)$ and as such induces a closure operator D on $L(C)$ [1].

LEMMA 1. *Let $DL(C)$ be finite. If M is a join-sublattice of $DL(C)$ such that $O_{DL(C)}$ is in M, then M is isomorphic to $DL(C')$ for some subset C' of C.*

Here $C' = \{p: p$ is an atom of $L(C)$ and $D(p)$ is in $M\}$.

The standard "hereditary set" construction can be modified to give the following result.

LEMMA 2. *If L is a lattice with $\#L = n + 1$, then L is isomorphic to a join-sublattice of 2^n containing* 0.

THEOREM. *If L is a finite lattice then there is a finite-dimensional convex set C such that L is isomorphic to $DL(C)$.*

Since 2^n is the face lattice of a simplex, the result follows from Lemma 1.

The set C obtained here is of large dimension and highly nonclosed. By imposing additional conditions on L, we can reduce the size of C. For example if L is coatomistic and has k coatoms, L is isomorphic to $DL(C)$ where C is a subset of the $(k - 1)$-dimensional simplex S and in this case the coatoms of L are in 1:1 correspondence with the facets of S.

BIBLIOGRAPHY

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