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ON A CLASS OF FOLIATIONS AND THE EVALUATION OF THEIR CHARACTERISTIC CLASSES

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This note discusses a class of foliations and a technique for evaluating the generalized Godbillon-Vey invariants on these foliations. The information obtained yields information about the cohomology of the Haefliger spaces $H^*(B\Gamma_n^r, \mathbf{R})$ and $H^*(F\Gamma_n^r, \mathbf{R})$, $r \geqslant 2$. The class of foliations contains examples which have been studied by others as well. In particular, the foliations examined in [KT2] and in [Y] are of this type.

Let $G^{\mathbf{C}}$ be a complex semisimple Lie group. There is a class of subgroups of $G^{\mathbf{C}}$ called parabolic subgroups, and the conjugacy classes of these subgroups are in 1-1 correspondence with subsets of the Dynkin diagram for $\mathcal{G}^{\mathbf{C}}$, the Lie algebra of $G^{\mathbf{C}}$ (see [S] for a more detailed exposition). If $P^{\mathbf{C}}$ is a parabolic subgroup then the Lie algebra $\mathcal{P}^{\mathbf{C}}$ of $P^{\mathbf{C}}$ can be written in the form $\mathcal{P}^{\mathbf{C}} = \mathcal{G}_1^{\mathbf{C}} \oplus \mathcal{T}_1^{\mathbf{C}} \oplus \mathcal{N}^{\mathbf{C}}$. Here $\mathcal{G}_1^{\mathbf{C}}$ is semisimple and has a Dynkin diagram obtained by removing the subset of vertices mentioned above from the Dynkin diagram for $\mathcal{G}^{\mathbf{C}}$. $\mathcal{T}_1^{\mathbf{C}}$ is an abelian subalgebra of $\mathcal{G}^{\mathbf{C}}$, $\mathcal{G}_1^{\mathbf{C}} \oplus \mathcal{T}_1^{\mathbf{C}}$ contains a Cartan subalgebra of $\mathcal{G}^{\mathbf{C}}$, and $\mathcal{N}^{\mathbf{C}}$ is a nilpotent subalgebra. In fact, $\mathcal{G}^{\mathbf{C}} = \mathcal{G}_1^{\mathbf{C}} \oplus \mathcal{T}_1^{\mathbf{C}} \oplus \mathcal{N}^{\mathbf{C}} \oplus \mathcal{N}^{-\mathbf{C}}$ where $\mathcal{N}^{-\mathbf{C}}$ is a nilpotent subalgebra isomorphic to $\mathcal{N}^{\mathbf{C}}$, and $[\mathcal{G}_1^{\mathbf{C}}, \mathcal{T}_1^{\mathbf{C}}] = 0$, $[\mathcal{G}_1^{\mathbf{C}} \oplus \mathcal{T}_1^{\mathbf{C}}, \mathcal{N}^{\mathbf{C}}] \subset \mathcal{N}^{\mathbf{C}}$, $[\mathcal{G}_1^{\mathbf{C}} \oplus \mathcal{T}_1^{\mathbf{C}}, \mathcal{N}^{-\mathbf{C}}] \subset \mathcal{N}^{-\mathbf{C}}$.

Now let $\mathcal{G}$ be a real form of $\mathcal{G}^{\mathbf{C}}$ such that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{T}_1 \oplus \mathcal{N} \oplus \mathcal{N}^-$ where $\mathcal{G}_1 = \mathcal{G}_1^{\mathbf{C}} \cap \mathcal{G}$, etc. Then $\mathcal{G}$ has a subalgebra $\mathcal{P} = \mathcal{G}_1 \oplus \mathcal{T}_1 \oplus \mathcal{N}$. If G has Lie algebra $\mathcal{G}$, then there is a discrete subgroup, $\Gamma \subset G$, with $\Gamma\backslash G$ a compact manifold (see [R]), and the left translates of $\mathcal{P}$ determine a foliation on $\Gamma\backslash G$. This is the foliation we study.

Let $W_n = P_n[c_1, \ldots, c_n] \otimes \Lambda^*(u_1, \ldots, u_n)$ be the cochain complex with $\deg c_i = 2i$, $\deg u_i = 2i - 1$, $dc_i = 0$, $du_i = c_i$. $P_n[c_1, \ldots, c_n]$ is the polynomial algebra in $c_1, \ldots, c_n$, truncated above deg $2n$ where n is the codimension of the above foliation. There is a map φ: $H^*(W_n, \mathbf{R}) \longrightarrow H^*(\Gamma\backslash G, \mathbf{R})$ giving characteristic classes for the foliation (see [BT] for the construction of φ). We analyse this map φ.

First note that, since a left invariant form on G induces a form in $\Lambda^*(\Gamma\backslash G, \mathbf{R})$, there is a map α: $H^*(\mathcal{G}, \mathbf{R}) \longrightarrow H^*(\Gamma\backslash G, \mathbf{R})$ where $H^*(\mathcal{G}, \mathbf{R})$ is the cohomology of the Lie algebra $\mathcal{G}$.

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PROPOSITION (SEE ALSO LEMMA 4.88 [KT1]). *There is a commutative diagram where the map* α *is injective.*

$$\begin{array}{ccc} H^*(G, \mathbf{R}) & \xrightarrow{\ \alpha\ } & H^*(\Gamma\backslash G, \mathbf{R}) \\ \uparrow \psi & \nearrow \varphi & \\ H^*(W_n, \mathbf{R}) & & \end{array}$$

We analyze the map ψ. Let $\overline{G}$ be a compact form of G. The key observation is that $H^*(G, \mathbf{C}) \approx H^*(\overline{G}, \mathbf{C})$. Let $\overline{G}_1$, $\overline{T}_1$ be the subgroups of $\overline{G}$ corresponding to $G_1^{\mathbf{C}}$, $T_1^{\mathbf{C}}$. We use the results of [BO] to describe $H^*(\overline{G}, \mathbf{C})$ in terms of the spectral sequence for the bundle $\overline{G}_1 \times \overline{T}_1 \longrightarrow \overline{G} \longrightarrow \overline{G}/\overline{G}_1 \times \overline{T}_1$. Specifically, let $x_1, \ldots, x_n \in H^*(\overline{G}_1 \times \overline{T}_1, \mathbf{C})$ be the primitive elements transgressing to $g_1, \ldots, g_n \in H^*(B_{\overline{G}_1 \times \overline{T}_1}, \mathbf{C}) = S$. Let ρ: $H^*(B_{\overline{G}}, \mathbf{C}) \longrightarrow S$ represent the map on characteristic classes induced by inclusion $\overline{G}_1 \times \overline{T}_1 \subset \overline{G}$. Let $I \subset S$ be the ideal generated by the image of ρ. Let $A = S/I \otimes H^*(\overline{G}_1 \times \overline{T}_1, \mathbf{C})$ be the complex with $d(1 \otimes x_i) = g_i \otimes 1$, $d(g_i \otimes 1) = 0$. Then $H^*(A, \mathbf{C}) \approx H^*(\overline{G}, \mathbf{C}) \approx H^*(G, \mathbf{C})$.

There is a homomorphism σ: $\overline{G}_1 \times \overline{T}_1 \longrightarrow \mathrm{Gl}(n, \mathbf{C})$ given by the adjoint representation of $\overline{G}_1 \times \overline{T}_1$ on the Lie algebra $N^{-\mathbf{C}}$. σ induces a map $\overline{\sigma}$: $H^*(B_{\mathrm{Gl}(n,\mathbf{C})}, \mathbf{C}) \longrightarrow S$. For each Chern class c_k we can choose an element ξ_k in the acyclic complex $S \otimes H^*(\overline{G}_1 \times \overline{T}_1, \mathbf{C})$, $d(g_i \otimes 1) = 0$, $d(1 \otimes x_i) = g_i \otimes 1$, with $d\xi_k = \overline{\sigma}(c_k) \otimes 1$. ξ_k determines an element $\overline{\xi}_k$ in A. Then we have a map ν: $W_n \longrightarrow A$, $\nu(c_k) = (\sqrt{-1})^k \overline{\sigma}(c_k) \otimes 1$, $\nu(u_k) = (\sqrt{-1})^k \overline{\xi}_k$.

THEOREM. *There is a commutative diagram where* γ *is induced by the coefficient map* $\mathbf{R} \subset \mathbf{C}$, *and* $\overline{\alpha}$ *is injective*

$$\begin{array}{ccc} H^*(A, \mathbf{C}) & \xrightarrow[\ \overline{\alpha}\ ]{} & H^*(\Gamma\backslash G, \mathbf{C}) \\ \uparrow \nu^* & & \uparrow \gamma \\ H^*(W_n, \mathbf{R}) & \xrightarrow[\ \varphi\ ]{} & H^*(\Gamma\backslash G, \mathbf{R}) \end{array}$$

The power of this theorem stems from the fact that A is a finitely generated complex whose cohomology is an exterior algebra on the primitive elements of $\overline{G}$. Thus, given a cocycle in A, it is feasible to try to determine the class it lies in.

EXAMPLES. Let $G = \mathrm{sl}(n + k, \mathbf{R})$, $k < n$ or $k = n = 1$,

$$\begin{aligned} G_1 \oplus T_1 &\approx \mathrm{sl}(n, \mathbf{R}) \oplus \mathrm{sl}(k, \mathbf{R}) \oplus \mathbf{R} \\ &= \{\|a_{ij}\| \in \mathrm{sl}(n + k, \mathbf{R}) | a_{ij} = 0 \text{ for } i > k, j \leqslant k \text{ or } i \leqslant k, j > k\}, \end{aligned}$$

$$P = \{\|a_{ij}\| \in \mathrm{sl}(n + k, \mathbf{R}) | a_{ij} = 0 \text{ for } i > k, j \leqslant k\}.$$

Then in $H^*(\Gamma\backslash \mathrm{SL}(n + k, \mathbf{R}), \mathbf{R})$ (and thus in $H^*(F\Gamma^r_{nk}, \mathbf{R})$) the classes $c_1^{nk} u_1 \cdots u_k u_{i_1} \cdots u_{i_l}$ for all $k < i_1 < \cdots < i_k \leqslant n$ are nonzero and linearly independent (this includes the class $c_1^{nk}u_1 \cdots u_k$).

These results have been obtained by Kamber and Tondeur for the case when $k = 1$ and can be found in [KT1] and [KT2].

It is possible to obtain information about the independence of classes when c_1^{nk} is replaced by another monomial in $c_1, \ldots, c_{nk}$ by comparing examples for different values of n and k. For instance, by comparing the example $k = 2$, $n = q$ with the example $k = 1$, $n = 2q$ one obtains: For $q \neq 2$, in $H^*(F\Gamma_{2q}^r, R)$ the set of classes

$$\{c_1^{2q}u_1u_2u_{i_1} \cdots u_{i_l}, c_2c_1^{2q-2}u_1u_2u_{i_1} \cdots u_{i_l} | 2 < i_1 < \cdots < i_l \leq q\}$$

are linearly independent.

By examining foliations on $\Gamma\backslash G/K$, where K is a compact subgroup of P, analogous information for classes in $H^*(B\Gamma_n^r, R)$ is obtained. For example, in $H^*(B\Gamma_{2q}^r, R)$ the set of classes

$$\{c_1^{2q}u_1u_{i_1} \cdots u_{i_l}, c_2c_1^{2q-2}u_1u_{i_1} \cdots u_{i_l} | 1 < i_1 < \cdots < i_l \leq q \text{ and the } i_j \text{ are odd}\}$$

are linearly independent.

For these constructions, other examples, and a more detailed exposition, see [B].

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