BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, May 1977

# TOWARDS ALGEBRAIC COBORDISM

BY VICTOR SNAITH

Communicated by E. H. Brown, Jr., September 9, 1976

**Abstract.** A new description of cobordism is given and, by analogy, cobordism theories are defined for an arbitrary ring.

1. Let $A$ be a ring with a unit. A cohomology theory, $MA$, might reasonably be called "the algebraic cobordism of $A$" if

(i) geometry over $A$ gave rise to elements in $\pi_*(MA)$, and

(ii) the existence of Chern classes for $A$ induced a transformation of cohomology theories from $MA$ to the algebraic $K$-theory of $A$.

Below I sketch the construction of theories which often satisfy (i) and (ii). Details will appear in [2], [3].

Let $X$ be a homotopy associative and commutative $H$-space. Let $T \subset \pi_*^S(X)$ be a finite subset of homogeneous elements. To this data is associated a periodic, commutative ring spectrum $X(T)$. $X(T)^*$ is the associated cohomology theory. For example, when $X = BU$ and $T$ consists of the generator $B \in \pi_2(BU)$, then $X(T)_{2k} = \Sigma^2 BU$ and $\epsilon_{2k}\colon \Sigma^2 X(T)_{2k} \longrightarrow X(T)_{2k+2}$ is equal to

$$\Sigma^2(\Sigma^2 BU) \xrightarrow{h} \Sigma^2(S^2 \times BU) \xrightarrow{\Sigma^2(B \oplus \mathrm{id})} \Sigma^2(BU).$$

Here $h$ is a Hopf construction and "id" is the identity map of $BU$.

When $X = BGLA^+$ for a ring $A$ and $T \subset \pi_*^S(BGLA^+)$, $X(T)^*$ is called the *algebraic cobordism of $A$ associated with $T$*. The terminology is motivated by (a)–(c) of the following result:

THEOREM 1.1. *Suppose* $\dim Y < \infty$; *then*:

(a) $BU(T)^0(Y) \simeq MU^{2*}(Y)$ *if* $T = \langle$*generator of* $\pi_2(BU)\rangle$;

(b) $BSp(T)^0(Y) \simeq MSp^{4*}(Y)$ *if* $T = \langle$*generator of* $\pi_4(BSp)\rangle$;

(c) $BO(T)^0(Y) \simeq MO^*(Y)$ *if* $T = \langle$*generator of* $\pi_1(BO)\rangle$;

(d) *if $F$ is a finite field and $T$ is a subset of $K_*(F)$ then* $BGLF^+(T)^0(Y) \simeq 0$;

(e) *if* $T = \langle$*generator of* $K_1(Z)\rangle$ *then* $BGLZ^+(T)^0(Y)$ *in general is a nontrivial group in which each element is of order* 2.

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*AMS (MOS) subject classifications* (1970). Primary 57A70, 55B15, 55F99; Secondary 18F25.

*Key words and phrases.* Cobordism, algebraic corbordism, $K$-theory, algebraic $K$-theory, $S$-equivalence.

Theorem 1.1 relates $K$-theory and cobordism very satisfactorily. For example, Adams operations in $KU^*$ induce Adams operations in $MU^*$ while Adams idempotents in $KU^*$ induce Adams idempotents in $MU^*$.

The starting point for Theorem 1.1 is the following:

THEOREM 1.2. *If* $1 \leqslant n \leqslant \infty$ *there exist stable equivalences*

(i) $BU(n) = \bigvee_{1=k}^{n} MU(k)$,

(ii) $BSp(n) = \bigvee_{1=k}^{n} MSp(k)$,

(iii) $BO(2n) = \bigvee_{1=k}^{n} BO(2k)/BO(2k-2)$ *and*

(iv) $BSO(2n+1) = \bigvee_{1=k}^{n} BSO(2k+1)/BSO(2k-1)$ *when localised away from* 2.

2.1. SKETCH OF PROOF OF THEOREM 1.2. The Becker-Gottlieb transfer is used to embed each classifying space, as a filtered space, into $QW = \varinjlim \Omega^n \Sigma^n W$ for suitable $W$. For example $BU$ is embedded in $QBU(1)$. The decompositions then follow from the decomposition theorem of [1].

2.2. SKETCH OF THEOREM 1.1. Consider the unitary example. Then

$$BU(T)^0(Y) = \varinjlim_N \{\Sigma^{2N} Y, BU\}$$

where $\{\ ,\ \}$ means homotopy classes of $S$-maps. Hence, by Theorem 1.2, if $\dim Y \leqslant 4t$

$$BU(T)^0(Y) \simeq \varinjlim_M \bigoplus_{t+M<k} \{\Sigma^{4M} Y, MU(k)\} \oplus \prod_{t-M \leqslant l} MU^{2l}(Y). \tag{2.3}$$

A careful study of the $S$-equivalences of Theorem 1.2 and some obstruction theory shows that only the cobordism part of (2.3) remains in the limit.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA