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THE MULTIPLICITY PROBLEM FOR 4-DIMENSIONAL SOLVMANIFOLDS

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Let N_3 be the 3-dimensional Heisenberg group whose underlying manifold is $\mathbf{R}^3$ and whose group multiplication is $(\xi, t)(\eta, z) = (\xi + \eta, t + z + \frac{1}{2}(yu - xv))$ where $\xi = (x, y), \eta = (u, v) \in \mathbf{R}^2$ and $t, z \in \mathbf{R}$. Every $\sigma \in GL_2(R)$ defines an automorphism of N_3 by the rule $\sigma(\xi, t) = (\sigma\xi, \det \sigma \cdot t)$. Let Δ be the subgroup of $GL_2(\mathbf{R})$ which maps the integer lattice Γ of N_3 onto itself. For $\sigma \in \Delta$ set $S\sigma = N_3 \rtimes \sigma(t)$, $\Gamma\sigma = \Gamma \rtimes gp(\sigma)$ where $gp(\sigma)$ is the group generated by σ and $\sigma(t)$ is the 1-parameter subgroup through σ. By [2] the analysis of the right regular representation R of $S\sigma$ on $L^2(\Gamma\sigma\backslash S\sigma)$ reduces to an analysis of the unitary operator $T\sigma$: $F \longrightarrow F \circ \sigma$ where $F \in L^2(\Gamma\backslash N_3)$. Denote again by R the right regular representation of N_3 on $L^2(\Gamma\backslash N_3)$. Then

$$L^2(\Gamma\backslash N_3) = \sum \oplus H_n$$

where $F \in H_n$ iff $R(0, z)F = e^{2\pi i n z}F$. Each H_n is R-invariant, the multiplicity of R restricted to H_n is $|n|$ and $T\sigma H_n = H_n$. We restrict for convenience our attention to $T\sigma$ restricted to H_n, $n \geqslant 1$. Let L denote the left regular representation of N_3.

Let $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $T\omega\theta_n = \theta_n$ is the space of n-degree "theta functions" in H_n of period i (see [1]). Set

$$\psi_1 = e^{2\pi i z} e^{\pi i x y} \sum_{l \in Z} e^{-\pi(g+l)^2} e^{2\pi i l x},$$
$$\psi_2 = L(1/2, 1/2, 0)\psi_1^2,$$
$$\psi_3 = L(1/2, 0, 0)\psi_1\, L(0, 1/2, 0)\psi_1\, L(1/2, 1/2, 0)\psi_1.$$

THEOREM 1. *The n functions $\psi_1^{n-j}\psi_2^{j/2}$, j even, $\psi_1^{n-j}\psi_2^{(j-3)/2}\psi_3$, j odd, $j = 0, 2, \ldots, n$ define an eigenbasis for θ_n relative to ω. The eigenvalues are the first n numbers in the infinite sequence*

$$1; -1, i, 1, -i, \ldots, -1, i, 1, -i.$$

From this result, the results on the "diamond group" $S\omega$ can be read off. This case using vastly different techniques appears in [2]. Also, this is equivalent to diagonalizing explicitly the finite Fourier transform

$$\omega^* = \frac{\sqrt{n}}{n} (e^{2\pi i (jk/n)}), \qquad 0 \leqslant j, k < n.$$

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Let $H_{n0} = H_1 \circ \binom{n\ 0}{0\ 1}$. Then $H_n = \Sigma_{j=0}^{n-1} \oplus L(0, j/n, 0)H_{n0}$. Write for $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbf{C}^n$, $|\alpha| = 1$, $T\alpha(g) = \Sigma_{j=0}^{n-1} \alpha_j L(0, j/n, 0)g$, $g \in H_{n0}$. Set $V\alpha = \{T\alpha(g): g \in H_{n0}\}$.

THEOREM 2. *Let $\sigma \in \Delta$. There exists an orthonormal basis σ^* and a unitary operator $g \longrightarrow g^\sigma$ of H_{n0} satisfying*

$$T\alpha(g) = T\sigma^*\alpha(g^\sigma), \qquad \alpha \in \mathbf{C}^n,\ |\alpha| = 1,\ g \in H_{n0}.$$

Up to unit multiple σ^* and $g \longrightarrow g^\sigma$ are uniquely determined by requiring the following.

(a) $\omega^* = (\sqrt{n}/n)(e^{2\pi i(jk/n)})$, $0 \leqslant j, k < n$.

$g \longrightarrow g^\omega$ is uniquely determined by requiring $(R_X g)^\omega = R_{\omega^{-1}x}(g^\omega)$.

(b) For $\tau = \binom{1\ 2}{0\ 1}$,

$$\tau^* = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & e^{2\pi i j^2/n} & \\ & & & \ddots \end{pmatrix}, \qquad j = 0, \ldots, n-1$$

and $g \longrightarrow g^\tau = g \circ \tau$.

THEOREM 3. *Let $\sigma \in \Delta$. There exists*

(1) *a unitary operator $g \longrightarrow g^\sigma$ of H_{n0},*

(2) *an orthonormal basis α_j, $0 \leqslant j \leqslant n-1$ of $\mathbf{C}^n$,*

(3) *characters χ_j, $0 \leqslant j \leqslant n-1$ satisfying*

$$T\alpha_j(g) \circ \sigma = \chi_j(\sigma) T\alpha_j(g^\sigma)$$

for all $g \in H_{n0}$.

Clearly $H_n = \Sigma_{j=0}^{n-1} V\alpha_j$ is a decomposition into R-invariant and irreducible subspaces which are σ-invariant.

REFERENCES

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