who have to cope with congestion in the real world often find simulation, or numerical evaluation, to be more useful tools in the computer age.

But blind simulation is a very expensive business, and the mathematician is still needed to provide insights rather than formulae. Information of a qualitative or approximate kind about stability, robustness, sensitivity, rates of convergence, may be just what the practical man needs, or if not will enable him to ask the right questions of his computer. There has been distinguished work along these lines, but much remains to be done; in particular the many-server queue still presents a formidable challenge.

Although the Russian school has been by no means isolated from Western developments, it has inevitably differed in emphasis, and an account from that viewpoint of the present state of the theory is valuable and stimulating. Borovkov has taken a very fundamental approach, fitting a wide variety of models into a general framework. Explicit formulae are kept in their place, and he usefully stresses the limiting results which justify robust approximations of real practical use.

He does not discuss the relevance of the theory to the real world, and the book is only (!) an authoritative synthesis of the underlying mathematics. It will be read, and with great profit, by mathematicians seeking uses for the powerful tools of random process theory. Will they be able to make any contribution, however indirect, to the world which does not read the learned journals? There is no doubt that modern telecommunications systems work better because of the achievements of Erlang and his successors, but some other applications of the theory have been less fruitful (largely because a queue is often a complex feedback mechanism). Perhaps one rather trite conclusion is that here, as in other areas of applied mathematics, mathematicians should direct their attention to questions to which someone, somewhere, wants to know the answer.

J. F. C. KINGMAN

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, May 1977

Matrix groups, by D. A. Suprunenko, Translations of Mathematical Monographs, vol. 45, American Mathematical Society, Providence, Rhode Island, 1976, viii + 252 pp., $31.20.

If I recall correctly, John Thompson prefaced his talk at the group theory symposium at the University of Illinois in 1967 with the remark "I believe in a heliocentric view of the universe, with the linear groups playing the role of the sun". Given the nature of the development of group theory in the past one hundred years, such a credo seems very appropriate. The earliest work in group theory was concerned mainly with permutation groups, the principal example of which was the Galois group acting on the roots of a polynomial equation. Interest in linear groups arose first in a geometric context. Boole and Cayley in the 1840's and 1850's initiated invariant theory, i.e. the study of rational functions of several variables that are invariant under various groups of linear or affine changes of variables. This work occupied the attention of many prominent mathematicians for many years; one major development was the annunciation of Klein's Erlanger Program, which stressed that geometric properties should be viewed as properties left invariant by suitable transformation groups. Several lines of inquiry arose in this geometric discussion; for instance, Lie developed his theory of continuous transformation groups, in order to study partial differential equations, while Bravais, Jordan and others studied the discrete groups that act on crystalline structures, in order to understand the geometry of molecules.

The book under review is not concerned with these concrete applications, so we will henceforth discuss linear groups from a discrete point of view. The notion of an abstract group developed in the middle years of the nineteenth century, and it is not clear who deserves credit for the definition. Cayley first defined finite groups in 1854 (by means of properties of the multiplication table); but, according to E. V. Huntington, the first explicit general axioms were given by Kronecker in 1870. In any case, the early practitioners of abstract group theory quickly realized that permutations and matrices are nicer than abstract group elements, since these concrete objects come equipped with various handy invariants (cycle structure, traces, invariant subspaces and the like). Therefore, representation theory arose as a primary tool in the theory of abstract groups. From Cayley's work, one knew that any group could be viewed as a permutation group on itself; by linearization, any finite group could be viewed as a group of matrices. (Throughout this discussion, we refer only to finite matrices; there is little known about groups of infinite matrices.) More generally, one discusses all representations, i.e. all homomorphisms of a given group into either a symmetric group or a linear group. The use of such representations leads to many results in group theory which are difficult or, perhaps, impossible to obtain otherwise. For instance, Burnside showed by this method that a finite group whose order is divisible only by two primes is solvable. It was not until a few years ago that a purely group-theoretic proof was found. To my knowledge, there is as yet no nonrepresentation-theoretic proof known of the theorem of Frobenius giving the structure of the class of finite groups that bear his name. To this day, every finite group theorist must have ordinary and modular representation theory in his arsenal. As for infinite groups, much less is known about representations (although there is a considerable body of knowledge about particular types of continuous representations of certain topological groups). Indeed, I know of no algorithm for determining whether a given group admits a finite-dimensional faithful (i.e. one-to-one) representation by linear transformations on a vector space over a skewfield. A recent result of Procesi shows that a periodic group which can be embedded in a linear group over a commutative ring must be locally finite. Hence, the finitely generated, infinite p-group discovered by Golod admits no such representation. On the other hand, L. Auslander and R. G. Swan have shown that any polycyclic group admits a faithful representation by integral matrices. More generally, Merzljakov has shown that the holomorph of a polycyclic group has a faithful integral representation (recall that the holomorph of a group G is the natural split extension of G by its automorphism group). A beautiful theorem of Malcev asserts that if a group G is the union of a directed system $\{H_\alpha\}$ of subgroups, each of which has a faithful representation of degree n (fixed) over a field F_α (variable), then there is a field F (obtained as an ultraproduct of the F_α's) over which G has a faithful representation of degree n.

This discussion leads to one of the three fundamental types of questions one asks about linear groups. Given a subgroup G of $GL(V)$ for some vector space V over a field F, one asks what effect the linearity of G has on its group-theoretic properties (e.g. If G is solvable, can the solvable length of G be estimated in terms of the dimension of V?), how G sits in $GL(V)$ (What is its normalizer, what are its conjugates ...?), and how G acts on V (Is G irreducible, i.e. without invariant subspaces, is V a direct sum of irreducible G-subspaces ...?).

The early work in the area concentrated on the third question, from the viewpoint of representations of finite groups. For example, we have Maschke's Theorem, which asserts that if the order of G is relatively prime to the characteristic of F, then every G-invariant subspace of V has a G-invariant complement, whence V is a direct sum of irreducible G-subspaces. A result showing the interplay between the second and third questions asserts that if F is an algebraically closed field, then G is irreducible if and only if the F-algebra of endomorphisms of V spanned by G is the full F-endomorphism ring of V, i.e. G contains $(\dim_F V)^2$ linearly independent matrices.

Let us now leave behind these remarks on representations of finite groups, and look at the structure of infinite linear groups. An excellent survey has recently been published in these pages, namely John Dixon's review of Wehrfritz' *Infinite linear groups*, which appeared in this Bulletin **80** (1974), 1071–1074. For the reader's convenience, we will recall some of that material, and add a few points not noted there.

One of the earliest purely abstract results on linear groups goes back to 1878, when Jordan showed that a finite group G of $n \times n$ complex matrices has an abelian normal subgroup A whose index in G is bounded by a certain function of n only. In 1911, Schur gave a generalization of this to infinite periodic groups of complex matrices. The case of finite groups in finite characteristic was handled by Brauer and Feit in 1966 (in this case, the bound depends not only on n, but also on the Sylow subgroups of G). Results of this sort are extremely valuable, since they give some control of the degrees of representations of finite simple groups.

The general subject of periodic linear groups has, in fact, been the source of much interest. In the process of proving his generalization of Jordan's theorem, Schur showed that any periodic complex linear group G is unitary relative to a suitable inner product, that such a group is completely reducible, and that if G is finitely generated, then it is finite. A result complementary to the last of these was Burnside's theorem of 1905, which shows that a linear group G (in characteristic zero) is finite if and only if it is of finite exponent. (In finite characteristic, finite exponent implies nilpotent-by-finite.) This theorem gave rise to the famous Burnside problem: Is a finitely generated group of finite exponent finite? This problem enjoyed much attention for about sixty years, and was recently resolved in the negative by Novikov and Adjan.

After these early efforts on periodic linear groups, there was a lengthy period during which little was done in the abstract theory of linear groups. There were isolated works, such as the studies of Malcev mentioned above on characterizing linear groups, and other papers of Malcev and Zassenhaus dealing with solvability. These efforts showed that a locally solvable linear group is solvable, with derived length bounded in terms of the degree. In fact, the group is necessarily nilpotent-by-abelian-by-finite. The bounds on derived length were improved later by Huppert and Dixon. Stronger results are available for solvable linear groups over the integers; namely, such groups are polycyclic, and the number of infinite cyclic factors in a normal series is bounded in terms of the degree. This result is essentially the converse of the Auslander-Swan result mentioned earlier. Suprunenko, in the late 1940's, began the work on nilpotent and solvable linear groups that has occupied him ever since.

However, most of the work on linear groups from 1910 to 1960 was somewhat specialized. There were extensive developments in the representation theory of finite groups, especially the modular (nonzero characteristic) case. Allied to these developments was a discussion of linear groups over finite fields as sources of simple groups. The most profound advance in this area came in 1955, when Chevalley showed how the classification of simple Lie groups could be used to construct finite simple groups. The process was later generalized by Steinberg, Hertzig, Ree and many others. Chevalley was also instrumental in a revival of interest in the subject of algebraic groups (the algebraic geometer's analogue of Lie groups), particularly the linear ones. Kolchin developed this theory further, with an eye toward the Picard-Vessiot theory of differential equations. The entire subject has blossomed recently in connection with the revolution in algebraic geometry; in particular, invariant theory has reappeared, with the emphasis now on determining the structure of orbit spaces as varieties. There was also some interest in properties of particular linear groups, such as the modular group. The emphasis here was strictly on applications to number theory and complex function theory. Lie theory was widely developed and applied in both mathematics and physics.

Starting around 1960, there was a rebirth of interest in many aspects of abstract linear group theory. Some recent results depend on new methods, while others are more in the spirit of hammer-and-tongs computation. To exemplify the first trend, we mention the method of finite approximation, first used by Malcev in 1940, and then strengthened by Platonov in 1966. The basic result asserts essentially that for any element z of a finitely generated linear group G, there is a homomorphism f from G to a linear group in finite characteristic, with the property that $z \notin \ker f$. From this, one may deduce for example that no finitely generated infinite linear group is simple, and that every finitely generated linear group has nilpotent Frattini subgroup. In another direction, there is a remarkable theorem proved by Tits in 1972: any linear group which is not solvable-by-locally finite contains a free subgroup on two generators. Many corollaries follow from this result. For instance, any Noetherian linear group is solvable-by-finite.

Along more computational lines are the results on solvable and nilpotent linear groups. The results here are usually somewhat technical, so we will mention just one structural result: If F is an algebraically closed field, then $GL(n,F)$ has only finitely many conjugacy classes of maximal solvable subgroups; the number of such classes is bounded by a function of n alone.

Along more technical lines, one has such things as catalogues of maximal solvable linear groups over finite fields. Kindred results are also available for nilpotent groups.

Let us turn to an examination of Suprunenko's book. The first chapter is an introduction to permutation groups, and it seems a bit out of place. A few of the ideas in it reappear in the linear group theory, and there is a strong analogy between permutation groups and linear groups. However, these points are not clearly made in the book, and it would probably have been preferable to intersperse this material in the body of the text, as needed. The second chapter contains gross generalities about the general linear group. It is here that a certain unevenness first appears. There is a lengthy and entirely elementary discussion of the matrix representation of a linear transformation. One presumes that potential readers of an advanced text in algebra have no need for such a discussion; but they might profit from a definition of algebraic groups, which are mentioned without explanation later on. The chapter also contains discussions of Dieudonné determinants and the normal subgroups of the full general linear group.

Chapter III is perhaps the most useful in the book, at least as a contribution to the expository literature. It contains Bass' description of the normal subgroups of the stable general linear group over a ring, and the Bass-Lazard-Serre-Mennicke results on the normal subgroups of the general and special linear groups over the integers. The importance of these theorems in algebraic K-theory is well understood.

Other chapters deal with reducibility, imprimitivity, solvable linear groups, periodic linear groups and nilpotent linear groups. The material covered here is generally done better in Wehrfritz' book. Many of the results mentioned earlier are either omitted or mentioned without proof. For example, one cannot find proofs of the Auslander-Swan theorem, nor of Tits' theorem in Suprunenko's book. The method of finite approximation is not discussed.

The book has other deficiencies as well. There are no exercises; the translation (done by the Israel Program for Scientific Translations) is adequate, but so stilted in places that it becomes hard to read; the typography and printing are in the muddy style which we have, unfortunately, come to expect in the publications of this Society. The book also contains bits of Russian chauvinism; for instance, the notion of wreath product is attributed to Kaloujnine, even though it was essentially known well over a century ago. In summary, the book is mediocre. The reader or library with a limited budget would do much better to purchase a copy of Wehrfritz.

W. H. GUSTAFSON

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, May 1977

Lecture notes on queueing systems, by Brian Conolly, John Wiley & Sons, New York, London, Sydney and Toronto, 1975, 176 pp., $9.95.

The theory of queues is a subarea of the field of stochastic models. It deals with the special stochastic processes which arise from the waiting lines made up of randomly arriving items (customers) which require processing by one or