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SPACES OF SMOOTH FUNCTIONS ON ANALYTIC SETS

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1. **Stability.** Let $X \supset Y$ be real analytic (or, more generally, closed semianalytic) subsets of R^n with $\dim X < n$, and let $M \subset N$ be submodules of $(C^\infty(R^n))^m$ obtained (as modules of global sections) on tensoring by $C^\infty(R^n)$ coherent real analytic subsheaves $\widetilde{M} \subset \widetilde{N}$ of $(\mathcal{O}(R^n))^m$, where $\mathcal{O}(R^n)$ denotes the sheaf of real analytic functions on R^n. Let $M(Y, X)$ (similarly for N) be the space of m-tuples ϕ of Taylor fields on X flat on Y such that at each point $x \in X$, ϕ_x is in the formal completion M_x of M at x. Let r: $N(Y, R^n) \longrightarrow N(Y, X)/M(Y, X) = P(Y, X)$ denote the restriction.

THEOREM 1. *There is a continuous* E: $P(Y, X) \longrightarrow N(Y, R^n)$ *such that* $rE = 1$.

Theorem 1 is proved using the approach of [1, Chapter 6], where it is shown that r: $N(Y, R^n) \longrightarrow N(Y, X)$ is onto, with modifications as in [5]; E is nonlinear.

The ideal I of analytic functions vanishing on a real analytic set need not be coherent, but using a suitable decomposition of X by (nonclosed) semianalytic subsets, on each of which I is globally generated, Theorem 1 can be applied to give, with $E(Y, X)$ denoting the space of smooth functions on X flat on Y.

THEOREM 2. *There is a continuous* E: $E(Y, X) \longrightarrow E(Y, R^n)$, *a right inverse for the restriction.*

J. Mather's proof ([2, in particular, p. 283 and following]), can then be applied to give

COROLLARY 1. *Infinitesimal stability implies stability for smooth proper mappings of* X *into a manifold.*

2. ***G*-manifolds.** Let G be a compact Lie group acting linearly on R^n and let ϕ: $R^n \longrightarrow R^m$ be a polynomial "Hilbert" map (i.e. ϕ induces a mapping from the polynomials on R^m onto the G invariant polynomials on R^n). Let $X \subseteq R^n$ be a G invariant analytic set and let $C_G^\infty(X)$ denote the space of G invariant smooth functions on X. The method of Theorem 1 (see [5]) gives

THEOREM 3. *There is a continuous* E: $C_G^\infty(X) \longrightarrow C^\infty(R^m)$ *such that* $\phi^*E = 1$.

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When X is a manifold this result has also been obtained by J. Mather [8].

COROLLARY 2. *G infinitesimal stability implies stability for proper smooth G invariant mappings of X into a G-manifold of finite orbit type.*

When X is a manifold this result has also been obtained by V. Poenaru [3].

3. **Immersions.** Let $X \subset R^n$ be a compact real analytic set and let $p \in X$. If the maximal ideal in the local ring of C^∞ functions on X at p has a minimal generating set of t elements, then by the Malgrange preparation theorem there is a smooth embedding ϕ of B^t into R^n with $\phi(0) = p$ and $Z = \phi(B^t)$ containing a neighbourhood V of p in X; also $\text{Im}(D\phi(0)) = T_p(X)$ is independent of ϕ, where $D\phi$ denotes the derivative of ϕ. $T(X) = \bigcup_{p\in X} T_p(X)$, the tangent space of X, is a subspace of $T(R^n)|_X$, the tangent space of R^n restricted to X; $T_p(X)$ is the fibre of $T(X)$ at p. $T(X)|_V$ is a subspace of $T(Z)|_V$. If K is a simplicial complex and M is a smooth manifold, let $L_K(T(X), T(M))$ denote the set of continuous maps F from K into the space of smooth fibrewise mappings from $T(X)$ into $T(M)$ which are linear embeddings on each fibre and such that for each $k \in K$ and $p \in X$ there are neighbourhoods $U \subset V$ of p and $W \subset K$ of k such that $F|_W$ restricted to $T(X)|_U$ may be extended to a continuous map from W into the smooth fibrewise linear bundle maps from $T(Z)|_U$ into $T(M)$. Two such maps are *homotopic* if they can be joined by an element of $L_{K\times I}(T(X), T(M))$. An *immersion* of X in M is a smooth mapping $f: X \longrightarrow M$ such that for each $p \in X$ there is a neighbourhood Q of p such that $f^*: C^\infty(M) \longrightarrow C^\infty(Q)$ is onto. If the space of immersions (with the Whitney topology as usual) of X in M is denoted by I_m and $[K, I_m]$ denotes the space of continuous maps from K into I_m, then

THEOREM 4. *The homotopy classes of* $L_K(T(X), T(M))$ *and* $[K, I_m]$ *are put in* 1-1 *correspondence by the derivative when* $\dim M - \text{Max}_{p\in X}(\dim T_p(X)) \geqslant 1$.

Theorem 4 may be proved by a reduction to the corresponding known result for manifolds.

4. **Diffeomorphisms.** Let $X \subseteq R^n$ be a closed semianalytic set with a semianalytic stratification such that for each $x \in X$ the dimension of the stratum through x and the dimension of the space of analytic tangent vectors of X at x coincide. Let $D(X)$ denote the space of smooth diffeomorphisms, let $I_s(X)$ denote the space of smooth isotopies of X *and let* $r(F) = F(1)$, *for each F belonging to* $I_s(X)$.

THEOREM 5. *There is a neighbourhood U of* 1 *in* $D(X)$ *and a continuous map* $E: U \longrightarrow I_s(X)$ *such that* $rE = 1$.

Theorem 5 is proved using the techniques of [5].

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