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THE HASSE NORM PRINCIPLE FOR ABELIAN EXTENSIONS OF NUMBER FIELDS

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1. Introduction. Let K be a finite extension of $\mathbf{Q}$, the field of rational numbers, and let L be a finite abelian extension of K. We say that the Hasse norm principle is valid for L/K if the following statement is true: each nonzero element of K is the norm of an element of L if and only if it is the norm of an element from each completion of L. It is well known that the Hasse norm principle is valid when L/K is cyclic, but the Hasse norm principle is not always valid for L/K when L/K is not cyclic (see [1, p. 199]). Our goal in this paper is to give an explicit, computable algorithm for determining whether the Hasse norm principle is valid for a given finite abelian extension L/K. Proofs will appear elsewhere. Before stating our results, we remark that Garbanati (see [2]) has obtained such an algorithm for certain finite abelian extensions L of $\mathbf{Q}$ of prime power degree, and Razar (see [3]) has also obtained some interesting results on the Hasse norm principle. Razar's results include results equivalent to Theorems 1 and 2 in the next section.

2. Main results.

THEOREM 1. *Let K be a finite extension of $\mathbf{Q}'$, and let L be a finite abelian extension of K. Let $l_1, \ldots, l_t$ be the distinct prime numbers dividing the order of* $\mathrm{Gal}(L/K)$, *and let L_i be the maximal l_i-extension of K contained in L, $1 \leq i \leq t$. Then the Hasse norm principle is valid for L/K if and only if the Hasse norm principle is valid for each L_i/K, $1 \leq i \leq t$.*

REMARK. Theorem 1 reduces the problem to the case where L/K is a finite abelian l-extension, where l is a prime number.

THEOREM 2. *Let K be a finite extension of $\mathbf{Q}$, and let L be a finite abelian l-extension of K, where l is a prime number. Let M be the maximal extension of K of exponent l contained in L. Then the Hasse norm principle is valid for L/K if and only if the Hasse norm principle is valid for M/K.*

REMARK. Theorem 2 reduces the problem to the case where L/K is a finite abelian l-extension with exponent l.

THEOREM 3. *Let K be a finite extension of $\mathbf{Q}$, and let L be a finite abelian*

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extension of K with exponent l, where l is a prime number. Let $G = \mathrm{Gal}(L/K)$, and let X be the group of characters of G. If $[L: K] = l^r$, let $\chi_1, \ldots, \chi_r$ be a basis for X over $\mathbf{F}_l$, where $\mathbf{F}_l$ is the finite field of l elements. For each nonarchimedean prime v of K which ramifies in L/K, let K_v (resp., L_w) be the completion of K at v (resp., L at w, where w is a prime of L above v). Let $G^v = \mathrm{Gal}(L_w/K_v)$, and if $[L_w: K_v] = l^s$, let $g_1, \ldots, g_s$ be a basis for G^v over $\mathbf{F}_l$. (Note $G^v \subseteq G$.) Let

$$[\delta_{tu,\alpha\beta}]_v, \quad 1 \leqslant t < u \leqslant s, \quad 1 \leqslant \alpha < \beta \leqslant r$$

be the matrix over $\mathbf{F}_l$ with $s(s-1)/2$ rows and $r(r-1)/2$ columns whose entry $\delta_{tu,\alpha\beta}$ in the tu row and $\alpha\beta$ column satisfies

$$\zeta^{\delta_{tu,\alpha\beta}} = (\chi_\alpha \wedge \chi_\beta)(g_t \wedge g_u),$$

where ζ is a primitive lth root of unity, and "$\wedge$" is the exterior (or alternating) product. Let Δ be the matrix (over $\mathbf{F}_l$) whose rows consist of all the rows of the matrices $[\delta_{tu,\alpha\beta}]_v$ as v ranges over all nonarchimedean primes v of K which ramify in L/K. Then the Hasse norm principle is valid for L/K if and only if rank $\Delta = r(r-1)/2$.

REMARK. For convenience we order the $s(s-1)/2$ pairs of numbers tu lexicographically. Also we order the $r(r-1)/2$ pairs of numbers $\alpha\beta$ lexicographically.

REMARK. Rank $\Delta \leqslant r(r-1)/2$ since Δ has $r(r-1)/2$ columns. If L/K is cyclic, then $r = 1$, and the matrix Δ has zero columns. Then rank $\Delta = 0 = r(r-1)/2$, and the Hasse norm principle is valid for L/K.

To facilitate the computation of $\delta_{tu,\alpha\beta}$, we have the following result.

THEOREM 4. *Let notations and assumptions be the same as in Theorem 3. Let $K' = K(\zeta)$ and $L' = L(\zeta)$. Let $a_1, \ldots, a_r \in K'$ such that*

$$L' = K'(\sqrt[l]{a_1}, \ldots, \sqrt[l]{a_r}).$$

Next let N_{L_w/K_v} be the norm map from L_w to K_v. Let $b_1, \ldots, b_s \in K_v^*$ such that the image of b_i in $K_v^*/N_{L_w/K_v}(L_w^*)$ corresponds to g_i under the isomorphism $K_v^*/N_{L_w/K_v}(L_w^*) \cong G^v$ of local class field theory, where $K_v^* = K_v - \{0\}$ and $L_w^* = L_w - \{0\}$. Let $\gamma_{ij} \in \mathbf{F}_l$ be defined by

$$\zeta^{\gamma_{ij}} = (a_j, b_i)_{v'}, \quad 1 \leqslant i \leqslant s, \quad 1 \leqslant j \leqslant r.$$

Here $(\ ,\ )_{v'}$ denotes the lth power Hilbert symbol in $K'_{v'}$ (cf. [1, p. 351]), where v' is a prime of K' above v, and $K'_{v'}$ is the completion of K' at v'. Then $\delta_{tu,\alpha\beta} = \gamma_{t\alpha}\gamma_{u\beta} - \gamma_{u\alpha}\gamma_{t\beta}$.

REMARK. Let $\hat{L}_w$ be the maximal abelian extension of K_v of exponent l, and let $G^v_{\max} = \operatorname{Gal}(\hat{L}_w/K_v)$. Then G^v is isomorphic to a factor group of $G^v_{\max}$; i.e., $G^v \cong G^v_{\max}/G^w$, where $G^w = \operatorname{Gal}(\hat{L}_w/L_w)$. Now it is not actually necessary to determine G^v in order to compute rank Δ. Instead of G^v we can use $G^v_{\max}$, and then it is easy to find $b_1, \ldots, b_s$. Of course using $G^v_{\max}$ instead of G^v may increase the number of rows in the matrix Δ, but rank Δ remains the same. This is true because G^w corresponds to $N_{L_w/K_v}(L^*_w)/N_{\hat{L}_w/K_v}(\hat{L}^*_w)$ under the isomorphism $K^*_v/N_{\hat{L}_w/K_v}(\hat{L}^*_w) \cong G^v_{\max}$ of local class field theory, and $(a_1, \;)_{v'}, \ldots, (a_r, \;)_{v'}$ are trivial on $N_{L_w/K_v}(L^*_w)$.

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