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RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of new results of such immediate importance to a substantial set of mathematicians that their communication to the mathematical community cannot await the normal time of publication. Research Announcements are communicated to the Bulletin by members of the Publications Committees listed on the inside back cover; manuscripts should be sent directly to any one of them. Each communicator is limited to accepting not more than six announcements each year; announcements are limited to 100 typed lines of 65 spaces each.

[*Note.* The criteria described above are new; in the transition period they will not necessarily be met by the announcements appearing in this issue.]

PRIME PI-RINGS

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In this note we introduce and sketch a new method for studying the behavior of prime ideals in prime rings satisfying a polynomial identity. If R is a prime PI-ring of $\mathrm{pid}(R) = n$ then R has a ring of quotients, $Q(R)$, which is a central simple algebra of dimension n^2 over its center. Each element of R, thus, has a (reduced) characteristic polynomial when considered as an element of $Q(R)$. T_R (or, simply, T if there is no ambiguity) will denote the ring generated by the coefficients of the characteristic polynomials of the elements of R. Our principal interest is the ring TR generated by R and T. TR is a central extension of R with the same quotient ring and, in general, more closely linked to its center than R is to its center as shown in

THEOREM 1. (a) *TR is integral over T.*

(b) *If $R = A[x_1, \ldots, x_n]$ where A is a commutative Noetherian ring, then TR is a finite T-module and $T = A[t_1, \ldots, t_v]$.*

Theorem 1 follows from a theorem of Shirshov [4] and a result of independent interest:

THEOREM 2. *Let Ω be a commutative ring and R an Ω-algebra. Assume every prime homomorphic image of R satisfies a polynomial identity. If every element of R is a sum of elements integral over Ω, then R itself is integral over Ω.*

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Our prinicpal result and the means of passing from R to TR is

THEOREM 3. *Let R be a prime ring such that* $\mathrm{pid}(R) = n$. *Then there exists an ideal V of R such that $TV \subseteq V$ and if P is a prime containing V then* $\mathrm{pid}(R/P) < n$.

Theorem 3 extends and simplifies results of Razmyslov [3] who considered the case when R is an algebra over a field of characteristic 0, and a somewhat different T when the characteristic of R is not 0.

The relationship between many of the primes of R and many of those of TR follows from the following general proposition.

PROPOSITION 4. *Suppose $R \subseteq S$ are rings and there is an ideal V of S which is contained in R. Then if P is a prime ideal of S, $P \not\supseteq V$, $P \cap R$ is a prime ideal of R and every prime ideal of R not containing V is obtained this way. Furthermore, if $P_1 \not\supseteq P_2$ and $P_1 \not\supseteq V$ are primes in S then*

$$P_1 \cap R \not\supseteq P_2 \cap R.$$

We remark that for a prime PI-ring R given *any* prime $P \subseteq R$ there exists a nonzero prime Q in TR such that $Q \cap R \subseteq P$ and $Q \cap R$ is prime.

Decisive results can be obtained in the important case of generic matrices [2, p. 61]. Let $R_{(v,n)}$ be the ring of v $n \times n$ generic matrices and $\mathcal{R}_{(v,n)} = TR_{(v,n)}$. We have

THEOERM 5. (a) *If P is a prime in $R_{(v,n)}$, then there is a prime $\mathcal{P} \subseteq \mathcal{R}_{(v,n)}$ such that $\mathcal{P} \cap R_{(v,n)} = P$.*

(b) *If $\mathcal{U}$ is a prime ideal of $\mathcal{R}_{(v,n)}$ then* $\operatorname{rank} \mathcal{U} = \operatorname{rank}(\mathcal{U} \cap R_{(v,n)})$.

Examples show that this result does *not* hold for arbitrary finitely generated rings. Furthermore, "going-up" does not hold for the prime ideals of the rings $R_{(v,n)}$ and $\mathcal{R}_{(v,n)}$.

As an application of Theorem 5 we do have the principal ideal theorem:

THEOREM 6. *Let $R_{(m,n)} = k[X_1, \ldots, X_m]$ be the ring of m generic $n \times n$ matrices. If z is a central nonunit of $R_{(m,n)}$, then any prime minimal over z is rank* 1.

For Noetherian prime PI-rings Theorem 1 can be improved. Indeed, we obtain

THEOREM 7. *Let R be a prime* PI-*ring which satisfies the ascending chain on centrally-generated ideals. Then TR can be generated as an R-module by finitely many central elements and also satisfies ACC on centrally-generated ideals. Thus, if R is Noetherian TR is also Noetherian.*

As an application of Theorem 7 we consider Noetherian G-rings. A prime PI-ring is a G-ring if its quotient ring can be obtained by inverting a single central element. Generalizing the commutative case [1, p. 107] is

THEOREM 8. *If R is a Noetherian G-ring, then R has only finitely prime ideals each of which is maximal.*

Complete proofs and additional applications will appear elsewhere.

W. Schelter informs us that he has obtained some of these results independently.

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